Math 2250-4

Wednesday 2/9

HW for Wed 2/16

3.5 5, 7, 8, 13, 22, 23, 30, 31, 32, 33.
hand in also techology checks (Maple, Matlab etc): 22 : A^{-1}, and that $AA^{-1}=A^{-1}A=I$, 23

Exam 1 Thurs!

3.6 2, 6, 17, 21, 22, 32
in 22 and 32 find x (x_1 in 32) with Cramer's rule. Then compute A^{-1} with the adjoint formula (page 210), as well as with the rref algorithm (better get same answer!). Finally, use A^{-1} to solve $A\vec{x}=\vec{b}$ and compare your Cramer's rule computations to the first components of your sol'ns.

- There's an interesting discussion on page 2 Tuesday about the general form of sol'ns to linear equations, and it ties back into DE's! We'll go over this in class.

Then, §3.4

- Matrix algebra

addition : If A and B are both $m \times n$, then

$$\text{entry}_{ij}(A+B) := a_{ij} + b_{ij}$$

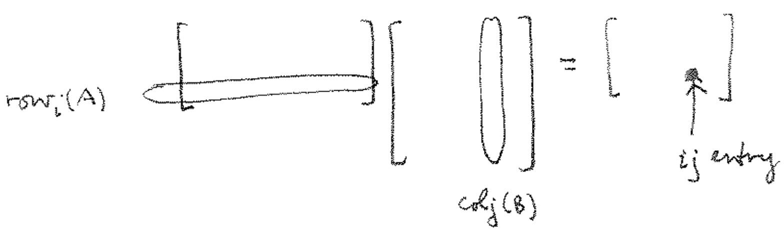
Scalar multiplication :

$$\text{entry}_{ij}(cA) := c a_{ij}$$

just like vector addition and scalar multiplication

matrix multiplication : [generalizes matrix times vector]

$$\text{entry}_{ij}(AB) = \text{row}_i(A) \cdot \text{col}_j(B)$$



so only works for

$$\big[A\big]_{m\times n}\big[B\big]_{n\times p} = \big[AB\big]_{m\times p}$$

examples :

Rules for this algebra

+ is commutative $A+B=B+A$

+ is associative $(A+B)+C=A+(B+C)$

scalar mult distributes over + $c(A+B)=cA+cB$

mult is associative $A(BC)=(AB)C$

mult distributes over + $A(B+C)=AB+AC$

$$(A+B)C=AC+BC$$

mult <u>not</u> commutative in general : DON'T EXPECT $AB=BA$!

<u>check properties</u>:

matrix algebra continued...

the $(n\times n)$ identity matrix $I := \begin{bmatrix} 1 & & & \\ 0 & 1 & & \bigcirc \\ 0 & 0 & \ddots & \\ & \bigcirc & & 1 \end{bmatrix}$ $\quad$ $entry_{jj}(I) = 1$

$entry_{ij}(I) = 0$ if $i \neq j$

$col_j(I) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow$ entry j $\quad := \vec{e}_j$

$row_i(I) = [0, 0, \ldots \underset{\uparrow \atop \text{entry } i}{1}, 0 \text{--} 0] = \vec{e}_i$ (written as a row vector)

$A_{m\times n} I_{n\times n} = A$

check: $entry_{ij}(A\,I) = row_i(A) \cdot col_j(I) = row_i(A) \cdot \vec{e}_j = a_{ij} = entry_{ij}(A)$

$I_{m\times m} A_{m\times n} = A$

check:

<u>Definition</u> $A_{n\times n}$ is <u>invertible</u> $:=$ there is a matrix B s.t. $AB = BA = I$

(and then we write A^{-1} for B)

[A can have at most 1 inverse, since if

$AB = BA = I$

and $AC = CA = I$

then $B(AC) = (BA)C$ $\quad$ associative prop

$B\,I = I\,C$

$B = C$!]

<u>Example</u> If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

Then, What good is A^{-1} ?

<u>Theorem</u> If A^{-1} exists, then the only solution to $A\vec{x} = \vec{b}$ is
$$\vec{x} = A^{-1}\vec{b}$$

Example continued:

Solve $\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 6 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \xrightarrow{\text{Theorem}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} -4 \\ 9/2 \end{bmatrix}$$

check ans:

check theorem:

But how did I know A^{-1} in that example?

<u>Ans</u>: I solved the matrix eqtn

$$A\,X = \text{Id}.$$

for $col_1(X)$: $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right]$

for $col_2(X)$: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$

for both columns at once:

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1+R_2 \end{array}\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+R_2 \\ \\ \end{array}\left[\begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{-\frac{R_2}{2}} \left[\begin{array}{cc|c|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$\uparrow$ col_1 $\quad \uparrow$ col_2

so $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

This is a general algorithm!

The matrix equation for $X = A^{-1}$ is

$$AX = I \qquad A_{n\times n},\ X_{n\times n},\ I_{n\times n}$$

Synthetically this is the n by $2n$ augmented matrix

$$A \,\vdots\, I$$ (where you are solving for all n columns at once!)

<u>If</u> this reduces to rref equal to

$$I \,\vdots\, B$$

Then $X = B$, so $AB = I$

[notice if you write $B \,\vdots\, I$ and do the row ops backwards you get $I \,\vdots\, A$, so also $BA = I$ is automatic].

<u>so</u>

<u>Thm</u> if $\text{rref}(A) = I$ then A^{-1} exists and can be found by algorithm above.

<u>Thm</u> if A^{-1} exists then unique sol'n to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

so, if A^{-1} exists we must also have $\text{rref}(A) = I$! ($\text{rref}(A) \neq I \Rightarrow$ never get unique sol'tns)

$$\boxed{A^{-1} \text{ exists iff } \text{rref}(A) = \text{id. iff } A\vec{x} = \vec{b} \text{ always has unique sol'tn}}$$

<u>example</u> $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-2R_1+R_2$, $-2R_1+R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

R_2+R_1, $-R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$2R_3+R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$-3R_2+R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

R_3+R_1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$