Math 2250-4
Monday Feb 7

On Friday we used Gaussian elimination to find all sol'ns to

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27 \end{aligned}$$

↓ synthetic version

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

↓ Gaussian elimination, stage 1 (working to the right and down)

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

row echelon form (not unique, i.e. a given matrix can have different row echelon forms)
1. all zero rows at bottom
2. leading non-zero entry in each row is strictly to the right of the previous rows

stands for

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 \quad + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

backsolve:

$x_5 = t$
$x_4 = 7 + 4t$
$x_3 = -3 - 2t$
$x_2 = s$
$x_1 = 10 - t - 2(7+4t) - 3(-3-2t) + 2s = 5 - 3t + 2s$

↓ Gaussian elimination, stage 2 (working up and to the left!)

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$

reduced row echelon form (unique!!)
1, 2, and
3. each non-zero row leads with a 1, called the leading 1
4. each column with one of the row leading 1's, has zeroes in every other column entry.

stands for

$$\begin{aligned} x_1 - 2x_2 \quad + 3x_5 &= 5 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}$$

backsolve

$x_5 = t$
$x_4 = 7 + 4t$
$x_3 = -3 - 2t$
$x_2 = s$
$x_1 = 5 - 3t + 2s$

Once you're comfortable with the reduced row echelon form algorithm use technology!

in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix} \quad ; s, t \in \mathbb{R}.$$

Math 2250–4
February 7, 2011
Maple linear algebra

```
> with(LinearAlgebra): #load Linear Algebra package, the newer
                       #of two Maple linear algebra libraries...its commands
                       #tend to begin with capital letters, in the old package
                       #linalg the commands are usually lower case
```

Example 1: Coefficient Matrix for last Friday's example, right hand side vector, augmented matrix:

```
> A := Matrix(3, 5, [1,-2, 3, 2, 1, 2,-4, 8, 3, 10, 3,-6, 10, 6, 5]);
                 #this is a 3 by 5 matrix, with entries written accross
                 #rows, from left to right and top to bottom.
  b := Vector([10, 7, 27]);
  B := <A|b>;  #augmented matrix
```

$$A := \begin{bmatrix} 1 & -2 & 3 & 2 & 1 \\ 2 & -4 & 8 & 3 & 10 \\ 3 & -6 & 10 & 6 & 5 \end{bmatrix}$$

$$b := \begin{bmatrix} 10 \\ 7 \\ 27 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix} \quad (1)$$

```
> ReducedRowEchelonForm(B);   #what we computed by hand on Friday
```

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \quad (2)$$

Maple will also directly solve a linear system – if there are infinitely many solutions than it may or may not be easy to recognize the form of Maple's solution as giving the same solutions that you write down by hand. Notice the way Maple chose to name its parameters.

```
> LinearSolve(A, b);   #solve the system Ax=b for x
```

$$\begin{bmatrix} 5 + 2\_t_2 - 3\_t_5 \\ \_t_2 \\ -3 - 2\_t_5 \\ 7 + 4\_t_5 \\ \_t_5 \end{bmatrix} \quad (3)$$

Example 2:
Coefficient matrix taken from problem #19, section 3.3, page 174. (You are assigned #20 to work by hand and check with technology).

```
> A:= Matrix(3,5,[2,7,-10,-19,13,1,3,-4,-8,6,1,0,2,1,3]);
     #the coefficient matrix
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \quad \textbf{(4)}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(5)}$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides (by working backwards from rref(A), actually). The three right hand sides are the columns of the matrix B:

```
> b1:=Vector([0,0,0]):
  b2:=Vector([7,0,0]):
  b3:=Vector([7,3,0]):
  C:=<A|b1|b2|b3>;  #triple-augmented matrix
```

$$C := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(6)}$$

We can consider all three linear systems at once by computing the reduced row echelon form of this triple–augmented matrix C. You'll have to put in the vertical line separating the coefficient matrix from the right hand sides by yourself.)

```
> ReducedRowEchelonForm(C);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{(7)}$$

From rref(C) you can read off the solutions to each of the three problems. Note that each column (from rref(A)) without a leading 1 in it gives you a free parameter in the solution. Write down the solutions to our three problems on the next page:

```
> ReducedRowEchelonForm(C);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (8)$$

Solutions to the three problems:

Important conceptual questions:

(1) Which of these three solutions could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix? Why?

(2) Linear systems in which right hand side vector equals zero are called homogeneous linear systems. Otherwise they are called inhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? We'll come back to this.

(3) The reduced row echelon form of a (non–augmented) matrix A can tell us a lot about the solution set to $Ax=b$. Deduce answers to all the questions below, and explain your answers.

```
> AA:=Matrix(2,5,[2, 7, -10, -19, 13,1, 3, -4, -8, 6]);
  ReducedRowEchelonForm(AA);
```

$$AA := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \end{bmatrix}$$

(9)

3a) Is the homogeneous problem $Ax=0$ always solvable?

3b) Is the inhomogeneous problem $Ax=b$ always solvable?

3c) When the inhomogeneous (or homogeneous) problem is solvable, how many solutions are there?

```
> B:=Matrix(3,2,[1,2,-1,3,4,2]);
  RREFB:=ReducedRowEchelonForm(B);
```

$$B := \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$RREFB := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{(10)}$$

4a) How many solutions to the homogeneous problem Bx=0?

4b) Is the inhomogeneous problem Bx=b always solvable?

4c) When the inhomogeneous proble is solvable, how many solutions does it have?

```
> C:=Matrix(4,4,[1,0,-1,1,22,-1,3,5,7,4,6,2,3,5,7,13]);
  RREFC:=ReducedRowEchelonForm(C);
```

$$C := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 22 & -1 & 3 & 5 \\ 7 & 4 & 6 & 2 \\ 3 & 5 & 7 & 13 \end{bmatrix}$$

$$RREFC := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

>

Square matrices with 1's down the diagonal (which runs from the upper left to lower right corner) are special. They are called identity matrices.

5a) How many solutions to the homogeneous problem Cx=0?

5b) Is the inhomogeneous problem Cx=b always solvable?

5c) How many solutions?

What are your general conclusions?

(1) What conditions on the reduced row echelon form of the matrix A guarantee that the homogeneous equation $Ax = 0$ has infinitely many solutions?

(2) What conditions on the dimensions of A always force infinitely many solutions to the homogeneous problem regardless of the individual entries of A?

(3) What conditions on the reduced row echelon form of A guarantee that solutions x to $Ax=b$ are always unique (if they exist)?

(4) If A is a square matrix ($m=n$), what can you say about the possible solutions to $Ax=b$ when

(4a) The reduced row echelon form of A is the identity matrix?

(4b) The reduced row echelon form of A is not the identity matrix?