Math 2250-4
Monday 2/28
↳ 4.4 - 4.5

How's your vocab?
vector space? subspace? basis? dimension?

Recall, the two easiest ways to get a subspace are as
- the solution set W to a homogeneous matrix equation
- the span of a collection of vectors

Today:
finding bases for the fundamental subspaces associated to a matrix $A_{m \times n}$:

Null(A):= the null space of A, i.e. the solution space for the homogeneous equation $Ax = 0$.
Col(A):= the span of the columns of A, i.e. the range of the function $T(x) = Ax$, in R^m.
Row(A):= the span of the rows of A, in the domain R^n of $T(x)$. Notice $Null(A)$ is the subspace of all vectors perpendicular ("orthogonal") to $Row(A)$.
(There's a fourth fundamental subspace, $Null(A^T)$, which is the subspace in the target space, of vectors orthogonal to the range $Col(A)$. We won't spend as much time of this subspace.)

basic question regarding the four fundamental subspaces: how can you deduce their dimensions and find bases using reduced row echelon form ideas?

We'll use the same matrix A in the following discussion. It's almost the one from Friday.

Exercise 1 (like Friday example)
Find a basis for Null(A)
Verify that your basis spans Null(A) and is linearly independent

with(LinearAlgebra) :

A := *Matrix*(4, 6, [1, 2, 0, 1, 1, 2,
2, 4, 1, 4, 1, 7,
1, 2, 1, 1, 2, 1,
2, 4, 0, 2, 2, 3]);

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -3 \end{bmatrix}$$

ReducedRowEchelonForm(*A*);

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 2 Based on the general features of exercise 1,

2a) Why does rref and backsolving with free parameters always identify a basis for the solution space of $A\vec{x}=\vec{0}$?

span:

linear ind:

2b) Give any matrix A, and its reduced row echelon form, how can you deduce the dimension of Null(A) with a quick count?

Exercise 3 Consider the column space of A, $Col(A) = span\{col_1(A), col_2(A), .. col_n(A)\}$

- Recall that any solution $\vec{x}$ to $A\vec{x}=\vec{0}$ is a dependency relation $x_1 col_1(A) + x_2 col_2(A) + \dots + x_n col_n(A) = 0$.
- But also, solutions to the homogeneous equation are the same for any two matrices differing by elementary row ops

conclusion: dependency (and independency) relations are the same for columns of A and columns of rref(A).

Use this reasoning to throw out dependent vectors (which does not shrink span), to get a basis for Col(A)

A := Matrix(4, 6, [1, 2, 0, 1, 1, 2,
2, 4, 1, 4, 1, 7,
1, 2, 1, 1, 2, 1,
2, 4, 0, 2, 2, 3]);

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -3 \end{bmatrix}$$

ReducedRowEchelonForm(A);

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3b) how can you deduce dim (Col(A)) by a quick count, using rref(A)?

Exercise 4 Let $W = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$. Consider the three "elementary set operations"

1) interchange v_ℓ and v_m
2) replace v_ℓ with cv_ℓ $c \neq 0$
3) replace v_ℓ with $v_\ell + cv_m$ $m \neq \ell$.

Show the new collection of vectors still spans W

Exercise 5 Find a "nicer" basis for $col(A)$ than you found in exercise 3, using the reduced column echelon form. Show how easy it is to express the #3 basis vectors in ~~the~~ terms of these nicer ones.

A;
ReducedRowEchelonForm(*A*);
Transpose(*ReducedRowEchelonForm*(*Transpose*(*A*))); #

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced column echelon form!

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 6: Find a basis for the Row space Row(A)
How can you deduce its dimension from rref(A)?

$$A \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -3 \end{bmatrix}$$

$$\mathrm{rref}(A) \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rcef}(A) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

called "nullity" of A

Summary: $\dim(\mathrm{Null}(A)) = \#$ columns in rref(A) <u>without</u> leading ones

$\dim(\mathrm{Col}(A)) = \#$ cols in rref(A) <u>with</u> leading 1's.

$\dim(\mathrm{Row}(A)) = \#$ rows in rref(A) <u>with</u> leading 1's

} this number is called the <u>rank</u> of A

and we get the fundamental theorem:

<u>Theorem</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A_{m\times n}\vec{x}$.

Then $\dim(\mathrm{range}(T)) + \dim(\text{nullspace of } T) = n = \dim(\text{domain space})$

$\mathrm{rank}(A) + \mathrm{nullity}(A) = n.$

Summary cont'd... **Key facts about independence, span, basis, dimension**
(in a fixed vector space V)

1) **Primary fact**: If a finite collection of vectors spans a vector space V, then any collection having a greater number of vectors is dependent

reason: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ span V, Let $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_{k+l}\}$ $(l>0)$ be a larger collection

We search for lin. comb. coeff's c_j s.t.

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_{k+l}\vec{w}_{k+l} = \vec{0}$$

Express each $\vec{w}_j$ as a linear combo of the spanning set:

$$c_1\begin{bmatrix} a_{11}\vec{v}_1 \\ +a_{21}\vec{v}_2 \\ + \\ \vdots \\ +a_{k1}\vec{v}_k \end{bmatrix} + c_2\begin{bmatrix} a_{12}\vec{v}_1 \\ +a_{22}\vec{v}_2 \\ \vdots \\ +a_{k2}\vec{v}_k \end{bmatrix} + \cdots + c_{k+l}\begin{bmatrix} a_{1\,k+l}\vec{v}_1 \\ +a_{2\,k+l}\vec{v}_2 \\ \vdots \\ +a_{k\,k+l}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\,\vec{v}_1 \\ +0\,\vec{v}_2 \\ + \vdots \\ +0\,\vec{v}_k \end{bmatrix}$$

equating coeff's for each $\vec{v}_i$, we get a dependency if we can find a $\vec{c} \neq \vec{0}$ so that

$$\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But the solution space to this homogeneous equation is at least l-dimensional, so non-zero sol'ns $\vec{c}$ exist. ∎

Logical consequences of 1:

2) If a finite collection of vectors in V is independent, no collection with fewer vectors can span

reason: logic! If a smaller set did span, the larger set would've been dependent! ∎

3) If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is a basis for V, then every other basis also consists of exactly k vectors. ∎

reason: fewer vectors can't span, more vectors would be dependent.

4) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ are independent, they span! (and are a basis)

reason: if they didn't span there would be a $\vec{v} \in V$ not in their span, so that the set $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k, \vec{v}\}$ would still be independent. This would violate 1! ∎

5) If $\dim V = k$ and if $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ span, then they're independent! (so are a basis)

reason: if they were dependent we could throw one of them away (i.e. a dependent one) without shrinking the span, so we would have $(k-1)$ vectors spanning V, in violation of 2! ∎