Math 2250-4
Friday Feb. 25
Vocabulary quiz! then §4.4.

HW for Friday March 4
4.4 1, 2, 3, 6 8, 9, 11, 13, 26
4.5 1, 12, 16, 17, 26, 27, 28
4.7 1, 2, 3 4, 5 6 7, 8 9, 10, 13, 15, 17, 21, 23, 25

(as in last week's HW, once you've set a problem up as a linear system you may use technology to compute rref, and just transcribe the result into HW before interpreting it.)

linear combination of $\vec{v}_1, \ldots \vec{v}_k$

span $\{\vec{v}_1, \ldots \vec{v}_k\}$

$\{\vec{v}_1, \ldots \vec{v}_k\}$ linearly independent
linearly dependent

V a vector space

W a subspace [not just any subset!]

§4.4

Def $\{\vec{v}_1, \ldots \vec{v}_k\}$ is a basis for V iff
(a) $V = \text{span}\{\vec{v}_1, \ldots \vec{v}_k\}$
(b) $\{\vec{v}_1, \ldots \vec{v}_k\}$ linearly independent

this is equivalent to saying that each $\vec{v} \in V$ can be uniquely expressed as
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$
in this case the linear combo coefficients are called the coords of $\vec{v}$ with respect to the basis $\{\vec{v}_1, \ldots \vec{v}_k\}$.

review
Example The standard basis for $\mathbb{R}^n$:
$\{\vec{e}_1, \ldots \vec{e}_n\}$ of $\mathbb{R}^n$

where $\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ entry j

(a) span: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$

(b) linearly independent: if $c_1\vec{e}_1 + c_2\vec{e}_2 + \ldots + c_n\vec{e}_n = \vec{0}$
then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ so all c_j's $= 0$

Exercise 1: On page 2 Wednesday, did we find a basis for $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } x + 2y - 3z = 0 \right\}$?
check!

after answering that question, finish Wednesday notes (Exercise 3 page 3, example page 4.)
Then return to today's notes ...

Recall, we figured out that

(a) more than n vectors in $\mathbb{R}^n$ are always linearly dependent
(b) less than n vectors in $\mathbb{R}^n$ cannot span $\mathbb{R}^n$

- so each basis of $\mathbb{R}^n$ has exactly n vectors
  and $\{\vec{v}_1, \ldots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ iff

$$A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right] \quad \text{satisfies } \operatorname{rref}(A) = I$$

<u>Def</u> For any vectorspace V, the dimension of V ($\dim(V)$) is defined ~~the~~ to be the number of vectors in every basis of V. } so, what are the dimensions of the subspaces we've considered today?

This definition only makes sense because

<u>Theorem</u> Every basis of V has the same number of vectors.

<u>Lemma</u>: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ span V. Then any collection $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_{k+\ell}\}$ $(\ell > 0)$ of more than k vectors in V is linearly dependent.

<u>proof of Lemma</u>:
we search for lin combo coef's s.t.

$$c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_{k+\ell} \vec{w}_{k+\ell} = 0.$$

Express each $\vec{w}_j$ as a linear combo of the v_i's:

$$c_1 \begin{bmatrix} a_{11}\vec{v}_1 \\ + a_{21}\vec{v}_2 \\ + \\ \vdots \\ + a_{k1}\vec{v}_k \end{bmatrix} + c_2 \begin{bmatrix} a_{12}\vec{v}_1 \\ + a_{22}\vec{v}_2 \\ + \\ \vdots \\ + a_{k2}\vec{v}_k \end{bmatrix} + \cdots + c_{k+\ell} \begin{bmatrix} a_{1\,k+\ell}\vec{v}_1 \\ + a_{2\,k+\ell}\vec{v}_2 \\ + \\ \vdots \\ + a_{k,k+\ell}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\vec{v}_1 \\ + 0\vec{v}_2 \\ + \\ \vdots \\ + 0\vec{v}_k \end{bmatrix}$$

We get a dependency if we can find $\vec{c} \neq \vec{0}$ so that

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But we can always do this because A has more columns than rows!

Lemma proven! ■

<u>proof of theorem</u>: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$, $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_{k+\ell}\}$ $\ell \geq 0$ be two bases of V, where the second collection has at least as many elements as the first

If $\ell \geq 1$, then the $\vec{w}_j$'s ~~are~~ would be dependent, i.e. not actually a basis.
Thus $\ell = 0$. ■

Exercise 2 Find a basis for the solution space to $A\vec{x}=\vec{0}$ for the matrix A below. Make sure to explain why your basis spans the solution space, and why it's linearly independent

```
> with(LinearAlgebra) :
> A := Matrix(4, 6, [1, 2, 0, 1, 1, 2,
                     2, 4, 1, 4, 1, 7,
                     -1, -2, 1, 1, -2, 1,
                     -2, -4, 0, -2, -2, -4]);
```

homogeneous equation

$$A := \left[\begin{array}{rrrrrr|r} 1 & 2 & 0 & 1 & 1 & 2 & 0 \\ 2 & 4 & 1 & 4 & 1 & 7 & 0 \\ -1 & -2 & 1 & 1 & -2 & 1 & 0 \\ -2 & -4 & 0 & -2 & -2 & -4 & 0 \end{array}\right]$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

```
>
```

Exercise 3 : Based on exercise 2,

3a) Why does our rref & backsolving with free parameters always yield a basis for the solution space to $A\vec{x}=\vec{0}$? (As happened in Exercises 1, 2)

span:

lin ind:

3b) What is dimension of solution space to $A\vec{x}=\vec{0}$ (i.e. how can you calculate the dimension from rref(A)?)