Math 2250-4
Wed. Feb. 23

Continue Tuesday notes ...

have you memorized the defintions of

linear combination of $\vec{v}_1, \vec{v}_2, .. \vec{v}_k$:

span $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$:

$\vec{v}_1, \vec{v}_2, .. \vec{v}_k$ are linearly dependent :

linearly independent :

- discuss the fact on page 2 Tuesday that $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ are linearly independent if and only if the linear combination coefficients of each $\vec{w}$ in span$\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ are unique.

- page 3 Tues: general reasoning about how many independent vectors it takes to span $\mathbb{R}^n$. (Such collections are called "bases" for $\mathbb{R}^n$.)

- p 4-5 Tues: vector spaces & subspaces (please bring notes!)

after page 4-5 Tuesday, on <u>vector spaces</u> and <u>subspaces</u>

<u>Exercise 1</u> Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } x+2y-3z=0 \right\}$ (you're used to saying "the plane $x+2y-3z=0$")

1a) Show W is a subspace by checking (α), (β),
i.e. closure with respect to addition and scalar multiplication

(α)

(β)

1a′) Is this a special case of the bullet point on page 5 Tuesday?
1b) Show W is a subspace by showing it is the span of 2 vectors in $\mathbb{R}^3$
hint: rref!

<u>Exercise 2</u> Is the plane $x+2y-3z=1$ a subspace? Hint: no! Explain!

Exercise 3 span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ must be a plane thru $\vec{0}$, in $\mathbb{R}^3$

Find the equation $ax + by + cz = 0$ for this plane

Ans: If $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ then we can solve

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ -c_1 \\ 2c_1 + c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{for } c_1, c_2$$

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ -1 & 0 & y \\ 2 & 1 & z \end{array}\right]$$

$$\begin{array}{l} -R_1 \\ R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & -y \\ 1 & 3 & x \\ 2 & 1 & z \end{array}\right]$$

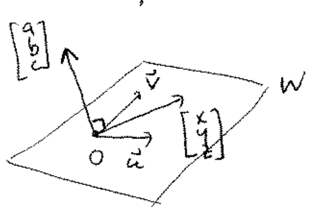
$$\begin{array}{l} \\ -R_1 + R_2 \\ -2R_1 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -y \\ 0 & 3 & x+y \\ 0 & 1 & z+2y \end{array}\right]$$

$$\begin{array}{l} \\ R_3 \\ R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -y \\ 0 & 1 & z+2y \\ 0 & 3 & x+y \end{array}\right]$$

$$\begin{array}{l} \\ \\ -3R_2 + R_3 \end{array} \left[\begin{array}{cc|c} 1 & 0 & -y \\ 0 & 1 & z+2y \\ 0 & 0 & x+y-3(z+2y) \end{array}\right] \quad x+y-3(z+2y) = x - 5y - 3z$$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ iff $\boxed{x - 5y - 3z = 0}$

another way for this special case:



$$ax + by + cz = 0$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = (-1, 5, 3)$$

$$-x + 5y + 3z = 0 \quad \checkmark$$

<u>example</u> Describe all subspaces of $\mathbb{R}^3$:

let W be such a subspace

(0) $\vec{0} \in W$.

that might be all!

if W has more elements, let $\vec{u} \in W$, $\vec{u} \neq \vec{0}$

then $\operatorname{span}(\vec{u}) \subset W$

$\uparrow$ "is a subset of"

(1) that might be all. In this case $W = \{t\vec{u},\ t \in \mathbb{R}\}$ is a <u>line thru $\vec{0}$</u>

if W has more elts, let $\vec{v} \in W$, $\vec{v} \notin \operatorname{span}\vec{u}$.

then $\{\vec{u}, \vec{v}\}$ are linearly independent: (if $c_1\vec{u} + c_2\vec{v} = \vec{0}$, $c_2 \neq 0 \Rightarrow \vec{v} \in \operatorname{span}\vec{u}$, so $c_2 = 0$. then $\vec{u} \neq 0 \Rightarrow c_1 = 0$)

and $\operatorname{span}\{\vec{u}, \vec{v}\} \subset W$.

(2) that might be all. In this case $W = \{t\vec{u} + s\vec{v},\ s, t \in \mathbb{R}\}$ is a plane.

See last example;

$$\left[\begin{array}{c|c|c} \vec{u} & \vec{v} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & \sim \\ 0 & 1 & \sim \\ 0 & 0 & ax+by+cz \end{array}\right]$$

if that's not all, then let $\vec{w} \in W$, $\vec{w} \notin \operatorname{span}\{\vec{u}, \vec{v}\}$.

Then

$$\left[\begin{array}{c|c|c} u & v & w \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(otherwise ~~either $\vec{v}$ and $\vec{u}$ are dependent or~~ $\vec{w}$ is a l.c. of $\vec{u}, \vec{v}$).

Thus $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$

(3). so $W = \mathbb{R}^3$

<u>Def</u> A <u>basis</u> for the vector space V is a linearly independent set $\{\vec{v}_1, \ldots \vec{v}_k\}$ in V which also spans V

<u>Def</u> The number of elements in a (any!) basis for V is the <u>dimension</u> of V.

We exhibited the dimension 0, 1, 2, 3 subspaces of $\mathbb{R}^3$.