Math 2250-4
Friday Feb. 18 (Replaces Wednesday notes)

§4.1-4.3

HW for Friday Feb. 25
4.1 1, 7, 9, 10, 15, 16, 19, 22, 25, 26, 31, 33
4.2 5, 6, 9, 15, 18, 24, 27, 29
4.3 1, 3, 6, 9, 10, 16, 17, 18, 23, 25
Please feel free to use technology to compute rref in any of this week's problems.

A lot of this chapter has to do with re-interpreting $A\vec{x}$ in linear combination form:

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}}_{\text{chapter 3}} = \underbrace{x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\text{chapter 4}}$$

linear combination form of $A\vec{x}$ expresses $A\vec{x}$ as a sum of multiples of the columns of A.

the language we will use:

If $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is a collection of vectors (say in $\mathbb{R}^n$) then $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ means

$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ for some choice of the scalar (numbers) $c_1, c_2, \dots c_n$. These numbers are called the linear combination coefficients

the span of $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is the collection of all $\vec{w}$'s which are linear combinations of $\vec{v}_1, \dots \vec{v}_k$.

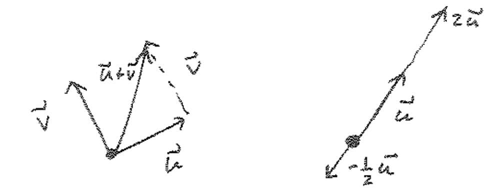
Example $A\vec{x} = x_1 col_1(A) + x_2 col_2(A) + \dots + x_n col_n(A)$ is always a linear combination of the columns of A

notice, the span of the columns of A is exactly the range of the function $L(\vec{x}) := A\vec{x}$
$L: \mathbb{R}^n \to \mathbb{R}^m$

Exercise 1 (to review geometric meaning of vector addition & scalar multiplication in $\mathbb{R}^n$, and to interpret linear combination problems geometrically.)

Can you get to $\begin{bmatrix}1\\8\end{bmatrix}$ in $\mathbb{R}^2$

from the origin, moving only in the $\begin{bmatrix}3\\-2\end{bmatrix}$ and $\begin{bmatrix}5\\1\end{bmatrix}$ directions?



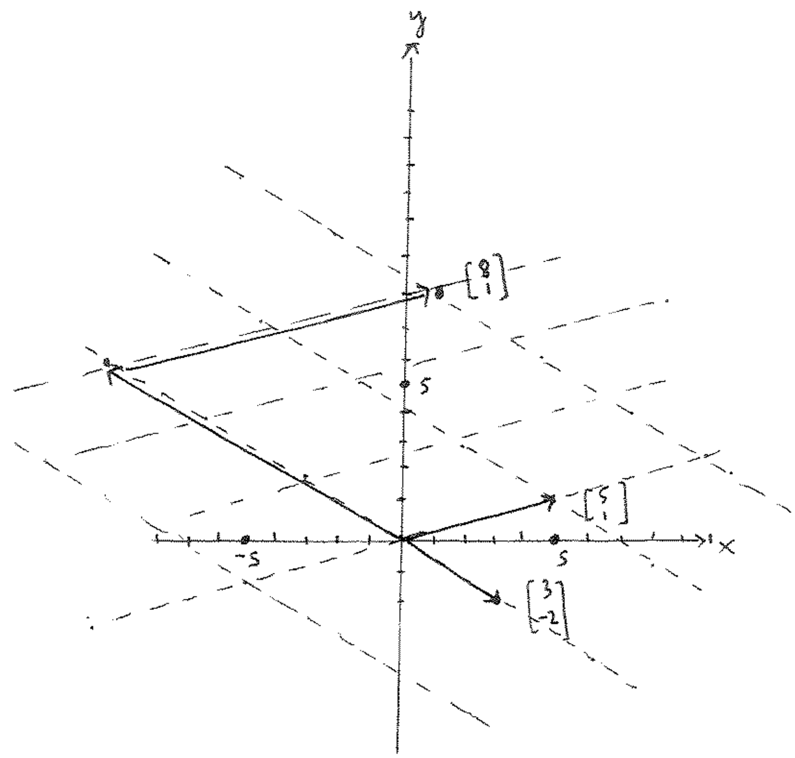
We seek linear combo coefficients s, t, so

$$s\begin{bmatrix}3\\-2\end{bmatrix}+t\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$

$$\begin{bmatrix}3&5\\-2&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$

$$\begin{bmatrix}s\\t\end{bmatrix}=\frac{1}{13}\begin{bmatrix}1&-5\\2&3\end{bmatrix}\begin{bmatrix}1\\8\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-39\\26\end{bmatrix}=\begin{bmatrix}-3\\2\end{bmatrix}$$



1b) Is $\operatorname{span}\left\{\begin{bmatrix}3\\-2\end{bmatrix},\begin{bmatrix}5\\1\end{bmatrix}\right\}=\mathbb{R}^2$

i.e. can we get anywhere in $\mathbb{R}^2$ from the origin, via linear combinations of these two vectors?

1c) If $s\begin{bmatrix}3\\-2\end{bmatrix}+t\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

are the linear combo coeffts unique?

Answer 1b), 1c) in terms of properties of the matrix $\begin{bmatrix}3&5\\-2&1\end{bmatrix}$.

More language!

If $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ are vectors in $\mathbb{R}^n$ that span $\mathbb{R}^n$ and so that whenever $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{b}$, the coef's $\{c_j\}_{j=1}^{k}$ are unique, then $\{\vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ are a basis for $\mathbb{R}^n$.

example $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

(so ~~are~~ is the "standard" basis $\left\{ \overset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$

Exercise 2 $\underset{\text{Physicist}}{\hat{i}} = \underset{\text{math person}}{\vec{e}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\hat{j} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\hat{k} = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are called the standard basis vectors for $\mathbb{R}^3$

a) Show $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ spans $\mathbb{R}^3$, and that linear combination coefficients are unique.
(so they're a basis).
(it's)

b) What geometric object is span$\{\vec{e}_1, \vec{e}_2\}$?

c) What geometric object is span$\{\vec{e}_1\}$?

Exercise 3 Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}$ be vectors in $\mathbb{R}^3$

3a) Is $\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$ a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

You might make use of the following Maple output:

i.e. find c_1, c_2, c_3 so that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

```
> with(LinearAlgebra) :
  AT := Matrix(3, 3, [2, 1, 1, -1, 1, -5, 4, -1, 11]) :
  A := Transpose(AT);
  b := Vector([6, 3, 3]) :
  Aaugb := ⟨A|b⟩;
  ReducedRowEchelonForm(Aaugb);
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$$A := \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 1 & -5 & 11 \end{bmatrix}$$

$$Aaugb := \begin{bmatrix} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3b) Are the linear combo coeff's in 3a) unique?

3c) Explain why $\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What geometric object is this span?

Hint: $\left[\begin{array}{ccc|c} 2 & -1 & 4 & x \\ 1 & 1 & -1 & y \\ 1 & -5 & 11 & z \end{array}\right] \xrightarrow[\text{(not rref)}]{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & y \\ 0 & 3 & -6 & -x+2y \\ 0 & 0 & 0 & -2x+3y+z \end{array}\right]$

Exercise 4 a) If you have 3 vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$, what matrix condition is equivalent to: $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$, and that the linear combo coeff's are unique?
(If this is the case, we call $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$.)

4b) Why does every basis of $\mathbb{R}^3$ have exactly 3 vectors?

to be continued...