Math 2250-4
Wed. Feb. 16

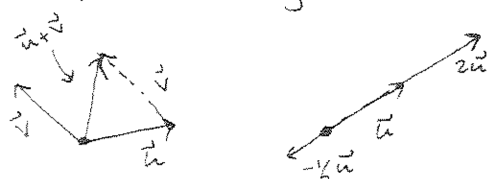
- introduce chapter 4 (in a way, it reviews Chptr 3)
- answer questions re exam & review sheet (last page Tues. notes)

Exam tomorrow!
Go to your section
(unless you've pre-arranged with Victor)
exam begins 5 minutes early, continues 5 minutes late, i.e. 60 minutes.

Chapter 4 Intro

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{chapter 3}}{=} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} \overset{\text{chapter 4}}{=} x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

and you will need to remember the geometric meaning of vector addition and scalar multiplication.



Exercise 1

How can you get to $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ by only moving in the $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ directions?

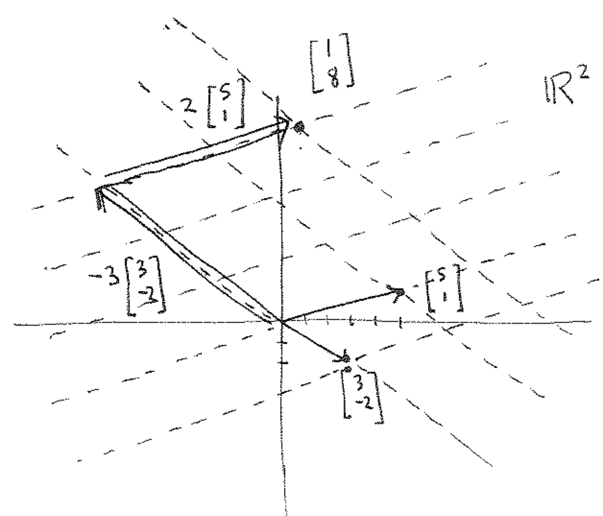
Solve

$$s \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} -39 \\ 26 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



the expression $s \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is called a <u>linear combination</u> of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

- Can we get anywhere in $\mathbb{R}^2$ via a linear combination of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$?
- Are the linear combination coefficients (s & t) unique, if $s \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

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the language we will use in Chapter 4 & beyond

If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is a collection of vectors (say in $\mathbb{R}^n$)
then $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ means

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

for some choice of the scalar coefficients $c_1, c_2, \ldots c_k$.
these numbers are called the linear combination coefficients.

the span of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is the collection of all $\vec{w}$'s which are linear combinations of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$

Exercise 2 We just showed $\begin{bmatrix}1\\8\end{bmatrix}$ is in the span of $\left\{\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}5\\1\end{bmatrix}\right\}$ since

$$\begin{bmatrix}1\\8\end{bmatrix} = -3\begin{bmatrix}3\\-2\end{bmatrix} + 2\begin{bmatrix}5\\1\end{bmatrix}.$$

- is every vector in $\mathbb{R}^2$ a linear combination of $\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}5\\1\end{bmatrix}$?
- what is span $\left\{\begin{bmatrix}3\\-2\end{bmatrix}, \begin{bmatrix}5\\1\end{bmatrix}\right\}$
- if $\vec{w} = s\begin{bmatrix}3\\-2\end{bmatrix} + t\begin{bmatrix}5\\1\end{bmatrix}$ are the linear combination coefficients s,t unique?

Remark if two vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$ and if each $\vec{w} \in \mathbb{R}^2$ can be expressed as a unique linear combination of these two vectors, then we call the vectors a basis for $\mathbb{R}^2$. How can we figure out whether two given vectors $\{\vec{v}_1, \vec{v}_2\}$ are a basis for $\mathbb{R}^2$?

Exercise 3 Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}$ be vectors in $\mathbb{R}^3$

3a) Is $\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$ a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

You might make use of the following Maple output:

i.e. find c_1, c_2, c_3 so that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

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> with(LinearAlgebra):
  AT := Matrix(3, 3, [2, 1, 1, -1, 1, -5, 4, -1, 11]):
  A := Transpose(AT);
  b := Vector([6, 3, 3]):
  Aaugb := <A|b>;
  ReducedRowEchelonForm(Aaugb);
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$$A := \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & -1 \\ 1 & -5 & 11 \end{bmatrix}$$

$$Aaugb := \begin{bmatrix} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Are the linear combo coeff's in 3a) unique?

c) Explain why $\mathrm{span}\{\vec{v}_1, \vec{v}_2\} = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What geometric object is this span?

Hint: $\left[\begin{array}{ccc|c} 2 & -1 & 4 & x \\ 1 & 1 & -1 & y \\ 1 & -5 & 11 & z \end{array}\right] \xrightarrow[\text{(not rref)}]{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & y \\ 0 & 3 & -6 & -x+2y \\ 0 & 0 & 0 & -2x+3y+z \end{array}\right]$

xercise 4 If you have 3 vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$, what matrix condition is equivalent to:
$\mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$, <u>and</u> that the linear combo coeff's are unique?
(If this is the case, we call $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a <u>basis</u> for $\mathbb{R}^3$.)