Math 2250-4
§3.6 Determinants
Tuesday Feb. 15.

a chance to go through 2008 midterm tonight 6-7:30 pm JTB 140
(you could also ask abt hw due Wed.)

recall: cofactor expansions for det
det and elementary row ops.
$|A| \neq 0$ iff $\text{rref}(A) = I$ iff A^{-1} exists.

another magic formula
$$|AB| = |A||B|$$
(see Monday notes... <u>not</u> true that $|A+B| = |A| + |B|$).

today: magic formulas for A^{-1} when $|A| \neq 0$.

<u>need</u>: <u>transpose</u> of a matrix (doesn't have to be a square matrix.)

<u>Def</u> If $B_{m \times n} = [b_{ij}]$ then the <u>transpose</u> of B, B^T is defined by
$$\text{entry}_{ij}(B^T) := \text{entry}_{ji}(B) = b_{ji}$$
the effect of this is to switch rows & columns, since e.g.
$$\text{entry}_{ij}(B^T) = \text{entry}_j(\text{row}_i(B^T))$$
$$\shortparallel$$
$$\text{entry}_{ji}(B) = \text{entry}_j(\text{col}_i(B))$$

<u>example</u>
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Theorem Recall the cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

where the minor M_{ij} is the det of the $(n-1)\times(n-1)$ matrix obtained by deleting row i and col j of A

the adjoint matrix is the transpose of the cofactor matrix, $[C_{ij}] = \text{cof}(A)$.

i.e.

$$\text{Adj}(A) = \text{cof}(A)^T$$

then, when A^{-1} exists, it has formula

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$$

exercise 1: Show that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this theorem reproduces our "magic" formula

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

exercise 2 for our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $\text{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$, $|A| = 15$

What is A^{-1}?
Check ans!

$$\text{adj}(A) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

exercise 3

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \operatorname{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

3a) the (1,1) entry of

$[A][\operatorname{adj}(A)]$ is $15. = 1\cdot 5 + 2\cdot 2 - 1(-6)$

explain why this is det(A), expanded across first row

3b) the (2,1) entry of $[A][\operatorname{adj}(A)]$ is $0\cdot 5 + 3\cdot 2 + 1(-6) = 0$.

Notice, you used the same cofactors as in 3a).
What matrix (which uses 2 rows of A) is this the determinant of?

3c) the (3,2) entry of $[A][\operatorname{adj}(A)]$ is $2\cdot 0 + (-2)(3) + 1(6) = 0$

What matrix (which uses 2 rows of A) is this the det of?

If you understand 3a, b, c, completely, you have just realized why

$$(A)(\operatorname{adj}(A)) = \det(A)\, I$$ for every square matrix,

and so, why $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$ when $\det A \neq 0$.

precisely, $\text{entry}_{ii}\ A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{ij} = |A|$

$k \neq i$, $\text{entry}_{ik}\ A(\operatorname{adj}(A)) = \sum_{j=1}^{n} a_{ij} C_{kj} = \begin{vmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_i(A) \\ \vdots \\ \text{row}_i(A) \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \text{slot } k \\ \\ \leftarrow \text{slot } i \end{matrix} = 0.$

$\boxed{\text{this proves the } A^{-1} \text{ formula theorem!}}$

There's a related formula for solving for individual components of $\vec{x}$, when $A\vec{x} = \vec{b}$ has unique solutions:

Cramer's rule

let $\vec{x}$ solve $A\vec{x} = \vec{b}$ for invertible A.

then $$x_k = \frac{\left| C_1 \middle| C_2 \middle| \cdots \middle| b \middle| \cdots \middle| C_n \right|}{|A|}$$

numerator is det of matrix obtained from A by replacing ~~row~~ column k by $\vec{b}$.

proof:
$$\begin{aligned} x_k &= \text{entry}_k\left(A^{-1}\vec{b}\right) \\ &= \text{entry}_k\left(\frac{1}{|A|}\,\text{adj}(A)\vec{b}\right) \\ &= \frac{1}{|A|}\,\text{row}_k\left(\text{adj}(A)\right)\cdot\vec{b} \\ &= \frac{1}{|A|}\underbrace{\sum_{l=1}^{n} C_{lk}\,b_l} \end{aligned}$$

this is the expansion of the det in numerator of Cramer's rule, down the k^{th} column!

exercise 4 : Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

a) with Cramer's rule

b) with A^{-1}, using adjoint formula

Math 2250-4
Review sheet for exam 1, which is Thurs 2/17 in your section
problem session TBA
(Tuesday evening or Wed. p.m.)

Chapter 1 : Methods to solve certain 1^{st} order DE's
integration, for $\frac{dy}{dx} = f(x)$
position, veloc, accel, when accel is a fcn of t alone.
separable DE's
growth & decay (exponential)
populations, radioactive decay, Newton's law of cooling (with constant ambient temp.)
drug elimination, rumor propogation, disease spread.
NO TORRICELLI ON EXAM
linear 1^{st} order DE's
mixing problems
slope fields and phase diagrams to understand qualitative behavior of sol'ns
without knowing formula for solution fcn.
How to draw these, especially for autonomous DE's.

Chapter 2 : Applications in depth
population models: logistic, doomsday/extinction, harvesting logistic;
understand derivations, how to find solutions (by separating variables),
how to plot slope fields & phase portraits, how to find equilibrium
solutions and evaluate stability/unstability.
equilibrium sol'ns & stability for general 1^{st} order autonomous DE's.
acceleration-velocity models, especially linear drag (force proportional
to velocity.)
NO NUMERICAL METHODS (2.4-2.6) ON EXAM.

Chapter 3 : Linear systems and matrices
solving linear systems by creating the augmented matrix, using elementary
row ops to get rref, deducing solution by backsolving.
geometric meaning of linear systems in 2 or 3 variables
Matrix algebra : addition, scalar multiplication, matrix multiplication
What algebra rules hold, and which one(s) don't.

Matrix inverses
how to compute via row operations
how to solve linear systems with the inverse matrix,
if the inverse exists.

Determinants
how to compute with cofactors
how to compute with row ops
Cramer's rule
Adjoint formula for the inverse, esp. 2×2 & 3×3 cases.

Be able to ~~do~~
solve problems and
to explain key ideas