

①

Math 2250-4

§ 3.6 Determinants.

Recall, $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} := a_{11}a_{22} - a_{21}a_{12}$ (and we often write $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$).
 straight lines

Determinants for $n \times n$ matrices are defined recursively:

If $A_{n \times n} = [a_{ij}]$

$$\text{then } \det A := \sum_{j=1}^n a_{1j} (-1)^{1+j} M_{1j} = \sum_{j=1}^n a_{1j} C_{1j}$$

(expansion across row₁).

in general $\boxed{M_{ij}}$ is the ij minor,
 the determinant of the $(n-1) \times (n-1)$
 matrix obtained from A by deleting
 row _{i} (A) and col _{j} (A).

$C_{ij} = (-1)^{i+j} M_{ij}$ is
 called the ij cofactor of
 A .

Theorem (proof is in text appendix)

You can compute $\det A$ by expanding
 across any row or down any column

So

$$|A| = \sum_{j=1}^n a_{ij} C_{ij} \quad \text{expand across row } i$$

$$= \sum_{i=1}^n a_{ij} C_{ij} \quad \text{down column } j$$

Example (cont'd).

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Verify that the
cofactor matrix
is

$$[C_{ij}] = \begin{bmatrix} 5 & -2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

then check expansions across
several rows and columns.

what happens if choose different
rows of A & cofactor matrix
to take dot products? different
columns?

Example

$$\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 96 & \pi & 3 & 0 \\ 1221 & 34 & 17 & 4 \end{vmatrix}$$

Theorem: If A is upper or lower triangular
then $|A|$ is just the product of the diagonal
entries

Computational Shortcuts: (and important for understanding too!)

Effects of elementary row ops on determinants:

(analogous results hold for elementary column operations)

(1) swapping two rows changes the sign of the determinant

proof: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$

so true for 2×2 .

Now it is easy to check general case by induction

$n=3 \rightarrow$ expand across the row that wasn't swapped and use the $n=2$ result on each minor.

In general, for $A_{(n+1) \times (n+1)}$ expand across an unswapped row and use the (inductive) assumption that result is true for $n \times n$ matrices.

(1b) So, if 2 rows are equal, $\det = 0$

proof: let $\det(A) = x$

if we swap rows the new det is $-x$

but rows were the same, so had to get same det,

i.e. $-x = x \Rightarrow x = 0$.

(2) multiplying a single row by c , multiplies the det by c .

proof: if you multiplied $\text{row}_i(A)$ by c , expand new matrix det across row_i :

$$\begin{vmatrix} R_1 \\ R_2 \\ cR_i \\ R_n \end{vmatrix} = \sum_{j=1}^n (c a_{ij}) C_{ij} = c \sum_{j=1}^n a_{ij} C_{ij} = c \det A.$$

(3) Coolest property: replacing $\text{row}_i(A)$ with $\text{row}_i(A) + c \text{row}_k(A)$ Does Not change det!

proof: we'll expand across $\text{row}_i(A)$:

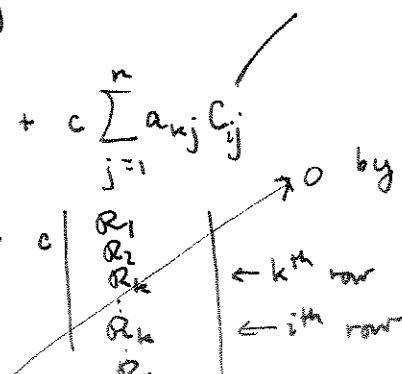
$$\text{row}_i \rightarrow \begin{vmatrix} R_1 \\ R_2 \\ R_i + cR_k \\ R_n \end{vmatrix} = \sum_{j=1}^n (a_{ij} + c a_{kj}) C_{ij}$$

$$= \sum_{j=1}^n a_{ij} C_{ij} + c \sum_{j=1}^n a_{kj} C_{ij}$$

$$= \det(A) + c \begin{vmatrix} R_1 \\ R_2 \\ R_k \\ R_k \\ R_n \end{vmatrix}$$

$\rightarrow 0$ by (1b)

Remark: analogous computations for columns



example

recompute $\det A$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

using elementary row operations

compute $|B|$ for $B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix}$

