Math 2250-4

§3.6 Determinants.

Recall, $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} := a_{11}a_{22} - a_{21}a_{12}$ (and we often write $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ).

straight lines

Determinants for $n \times n$ matrices are defined recursively:

If $A_{n\times n} = [a_{ij}]$

then $\det A := \sum_{j=1}^{n} a_{1j} (-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}$

(expansion across $row_1$).

in general $\boxed{M_{ij}}$ is the $ij$ minor, the determinant of the $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting $row_i(A)$ and $col_j(A)$.

$C_{ij} = (-1)^{i+j} M_{ij}$ is called the $ij$ cofactor of $A$.

Theorem (proof is in text appendix)

You can compute $\det A$ by expanding across any row or down any column

So

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{expand across } row_i$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{down column } j$$

Example (cont'd).

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

Verify that the cofactor matrix is $[C_{ij}] = \begin{bmatrix} 5 & -2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

then check expansions across several rows and columns.

what happens if choose <u>different</u> rows of A & cofactor matrix to take dot products? different columns?

## Example

$$\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 96 & \pi & 3 & 0 \\ 1221 & 34 & 17 & 4 \end{vmatrix}$$

**Theorem**: If $A$ is upper or lower triangular then $|A|$ is just the product of the diagonal entries

**Computational Shortcuts:** (and important for understanding too!)

Effects of elementary row ops on determinants:
(analogous results hold for elementary column operations)

(1) swapping two rows changes the sign of the determinant

proof: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ; $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$

so true for $2 \times 2$.

Now it is easy to check general case by <u>induction</u>

$n=3 \rightarrow$ expand across the row that wasn't swapped and use the $n=2$ result on each minor.

In general, for $A_{(n+1)\times(n+1)}$ expand across an unswapped row and use the (inductive) assumption that result is true for $n \times n$ matrices.

(1b) So, if 2 rows are equal, det $= 0$

<u>proof</u>: Let $\det(A) = x$
if we swap rows the new det is $-x$
but rows were the same, so had to get same det,
i.e. $-x = x \Rightarrow x = 0$.

(2) multiplying a <u>single</u> row by $c$, multiplies the det by $c$.

proof: if you multiplied $\text{row}_i(A)$ by $c$, expand new matrix det across $\text{row}_i$:

$$\begin{vmatrix} R_1 \\ R_2 \\ cR_i \\ \vdots \\ R_n \end{vmatrix} = \sum_{j=1}^{n} (c a_{ij}) C_{ij} = c \sum_{j=1}^{n} a_{ij} C_{ij} = c \det A.$$

(3) Coolest property: replacing $\text{row}_i(A)$ with $\text{row}_i(A) + c\,\text{row}_k(A)$ Does Not change det!

proof: we'll expand across $\text{row}_i(A)$:

$$\text{row}_i \rightarrow \begin{vmatrix} R_1 \\ R_2 \\ R_i + cR_k \end{vmatrix} = \sum_{j=1}^{n} (a_{ij} + c a_{kj}) C_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + c \underbrace{\sum_{j=1}^{n} a_{kj} C_{ij}}_{\rightarrow 0 \text{ by (1b)}}$$

$$= \det(A) + c \begin{array}{|c|l} R_1 & \\ R_2 & \\ R_k & \leftarrow k^{th} \text{ row} \\ \vdots & \\ R_k & \leftarrow i^{th} \text{ row} \\ R_n & \end{array}$$

<u>Remark</u>: analogous computations for columns

example

recompute $\det A$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

using elementary row operations

compute $|B|$ for $B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix}$

**Theorem** $\det A \neq 0$ iff $\mathrm{rref}(A) = I$ iff $A^{-1}$ exist

already know this equivalence

pf: start with $A$

do elementary row ops

- change sign if swap rows (original det is opposite of new det.)
- no change if replace $\mathrm{row}_i$ by $\mathrm{row}_i + c\,\mathrm{row}_k$ $k \neq i$
- if factor "c" out of row, original det is $c$ times det of what's left

rref(A)

if $\mathrm{rref}(A) = I$

$$|A| = \underbrace{c_1 c_2 \cdots c_N}\ \underbrace{|I|}_{1}$$

$c_i$'s are $\pm 1$, or other nonzero const's.

and $|A| = c_1 c_2 \cdots c_N \neq 0$!

if $\mathrm{rref}(A) \neq I$

rref(A); has a row of 0's.

$$|A| = c_1 c_2 \cdots c_N \begin{vmatrix} \text{NNN} \\ 0\,0\,0 \cdots 0 \end{vmatrix} = 0$$

**Thm** $\det(AB) = (\det A)(\det B)$ ← true, but not obvious:

$\det(A+B) \neq \det A + \det B$ !!!

$|A|$

rref

$\mathrm{rref}(A) = I$: $|A| = \underbrace{c_1 \cdots c_N}_{|A|} |I|$

$\mathrm{rref}(A) \neq I$: $c_1 \cdots c_N \begin{vmatrix} \text{NN} \\ 000 \end{vmatrix}$

$|AB|$

do indentical rowops to $AB$ → is same as doing row ops to $A$, then multiplying

$\mathrm{rref}(A) = I$:

$|AB| = c_1 \cdots c_N |IB|$

$= |A||B|$

$\mathrm{rref}(A) \neq I$:

$|AB|$

$= c_1 \cdots c_N \left|\begin{pmatrix} \text{NN} \\ 000 \end{pmatrix} B\right|$

$= c_1 \cdots c_N \begin{vmatrix} \text{NN} \\ 0000 \end{vmatrix}$

$= 0$

$= |A||B|$.