Name: Solutions

Student I.D. number..

Math 2250-4

Exam 2

March 31, 2011

The exam is closed book and closed note, but a scientific calculator is allowed and may be useful. Calculators which can do linear algebra or solve differential equations are not allowed. In order to receive partial credit or full credit all work must be shown. There are 100 points possible on the test, and the point values of each problem are indicated in the right margin. You may wish to ration your time accordingly. GOOD LUCK!

Score:	Possible
1________	30
2________	40
3________	20
4________	10
Total________	100

1) Here is a matrix A (on the left), and on the right is its reduced row echelon form:

$$A := \left[\begin{array}{ccccc} 1 & -2 & 1 & -1 & -1 \\ -2 & 4 & -2 & 2 & 2 \\ 1 & -2 & 0 & 2 & 1 \\ 3 & -6 & 2 & 2 & 1 \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \qquad reduced := \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$$

1a) Find all solutions to the homogeneous linear equation Ax=0. (8 points)

backsolve rref(A).

$x_1 = 2r + t$
$x_2 = r$
$x_3 = -t$
$x_4 = -t$
$x_5 = t$

in linear combo form

$$\vec{x} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

1b) Exhibit a basis for the solution space you found in part 1a). (4 points)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

1c) What is the dimension of this solution space? (3 points)

dim = # of vectors in any basis = 2

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \quad \vec{v}_5$

$\downarrow \quad \downarrow$

Here again are the matrix A and its reduced row echelon form:

$$A := \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ -2 & 4 & -2 & 2 & 2 \\ 1 & -2 & 0 & 2 & 1 \\ 3 & -6 & 2 & 2 & 1 \end{bmatrix} \qquad reduced := \begin{bmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1c) Express the fifth column of A as a linear combination of the preceding columns. (5 points)

since column dependencies $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_5\vec{v}_5 = \vec{0}$ are equivalent to homogeneous sols $A\vec{c} = \vec{0}$, they are preserved as you convert A to its rref

so, $\vec{v}_5 = -\vec{v}_1 + \vec{v}_3 + \vec{v}_4$ since the identical relation clearly holds for the corresponding columns of rref(A)

(check: $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+1-1 \\ 2-2+2 \\ -1+2 \\ -3+2+2 \end{bmatrix}$ ✓)

1d) Find a basis for the column space of A (the span of the columns of A). Justify why your basis satisfies the two necessary conditions for it to be a basis of the column space. (10 points)

removing dependent vectors from a collection does not shrink the span.

so $\text{span}\{\vec{v}_1, \underbrace{\vec{v}_2}_{=-2\vec{v}_1}, \vec{v}_3, \vec{v}_4, \underbrace{\vec{v}_5}_{=-\vec{v}_1+\vec{v}_3+\vec{v}_4}\} = \text{span}\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$

$\vec{v}_1, \vec{v}_3, \vec{v}_4$ are independent, because the corresponding cols of rref(A) are.

thus $\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}$ are a basis for the column space.

(other bases are possible.)

2) Consider the differential equation below, which could model a mechanical oscillation or electrical circuit problem.

$$x''(t) + 4\cdot x'(t) + 13\cdot x(t) = 0.$$

2a) If this differential equation was modeling an unforced damped mass-spring configuration such as we've studied, and if the mass was 2 *kg*, determine the damping coefficient and (Hooke's) spring constant. Include correct *mks* units. (6 points)

$1\ N = 1\ kg\,m/s^2$

$m x'' + c x' + k x = 0$

$m = 2$

$2x'' + 8x' + 26x = 0.$

$c = 8\ Ns/m$ (or kg/s)

$k = 26\ N/m$ (or kg/s^2)

2b) Find a basis for the solution space to this homogeneous linear differential equation. (10 points)

$x = e^{rt}$

$Lx = p(r)e^{rt} = (r^2 + 4r + 13)e^{rt} = 0$ for roots of $p(r)$

$(r+2)^2 + 9 = 0$

$(r+2+3i)(r+2-3i) = 0 \quad r = -2 \pm 3i$

(or quad. form. $r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -2 \pm 3i$)

$e^{(-2+3i)t} = e^{-2t}(\cos 3t + i \sin 3t)$

so

$x_H(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

basis: $\{e^{-2t}\cos 3t,\ e^{-2t}\sin 3t\}$

2c) What sort of damping is exhibited by solutions to this differential equation? (4 points)

underdamped.

2d) Use the solutions you found in (2b) and Chapter 5 techniques to solve the general initial value problem

$$x''(t) + 4\cdot x'(t) + 13\cdot x(t) = 0.$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

(10 points)

$x(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$x'(t) = c_1\left(-2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t\right) + c_2\left(-2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t\right)$

$x(0) = x_0 = c_1$

$x'(0) = v_0 = -2c_1 + 3c_2$

so $c_1 = x_0$

so $v_0 = -2x_0 + 3c_2$

$\cancel{v_0 + 2x_0 = 3c_2}$

$\frac{1}{3}(v_0 + 2x_0) = c_2$

$$\boxed{x(t) = x_0 e^{-2t}\cos 3t + \tfrac{1}{3}(v_0 + 2x_0)e^{-2t}\sin 3t}$$

(or $\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ \frac{1}{3}(2x_0 + v_0) \end{bmatrix}$)

2e) Re-solve the initial value problem in (2d), using Laplace transforms.

(10 points)

$s^2 X(s) - sx_0 - v_0 + 4\left(sX(s) - x_0\right) + 13X(s) = 0$

$X(s)\left(s^2 + 4s + 13\right) = sx_0 + v_0 + 4x_0 = (s+4)x_0 + v_0$

$X(s) = \dfrac{(s+4)x_0 + v_0}{s^2 + 4s + 13} = \dfrac{(s+4)x_0 + v_0}{(s+2)^2 + 9}$

$= \dfrac{(s+2)x_0}{(s+2)^2 + 9} + \dfrac{2x_0 + v_0}{(s+2)^2 + 9}$

$\swarrow \mathcal{L}^{-1}$

So $x(t) = x_0 e^{-2t}\sin 3t + \left(\dfrac{2x_0 + v_0}{3}\right)\underbrace{\mathcal{L}^{-1}\left(\dfrac{3}{(s+2)^2 + 9}\right)}_{e^{-2t}\sin 3t}$

✓

3) Consider the initial value problem below, which could arise as a forced mass-spring (or electrical circuit) problem.

$$x''(t) + 9\cdot x(t) = 5\cdot\cos(2\cdot t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

3a) Use Laplace transform techniques to solve this problem. (15 points)

$$s^2 X(s) - s x_0 - v_0 + 9 X(s) = 5 \frac{s}{s^2+4}$$

$$X(s)(s^2+9) = \frac{5s}{s^2+4} + s x_0 + v_0$$

$$X(s) = \frac{5s}{(s^2+9)(s^2+4)} + \frac{s x_0 + v_0}{s^2+9}$$

$$= 5s\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)\frac{1}{5} + \frac{s x_0 + v_0}{s^2+9}$$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9} + x_0 \frac{s}{s^2+9} + \frac{v_0}{s^2+9}$$

$$= \frac{s}{s^2+4} + (x_0 - 1)\frac{s}{s^2+9} + \frac{v_0}{3}\frac{3}{s^2+9}$$

$\mathcal{L}^{-1}$: $\boxed{x(t) = \cos 2t + (x_0-1)\cos 3t + \frac{v_0}{3}\sin 3t}$

(check: $x(0) = 1 + (x_0 - 1) = x_0$ ✓, $x'(0) = \frac{v_0}{3}\cdot 3 = v_0$ ✓).

3b) Do solutions to this differential equation exhibit resonance? Explain. (5 points)

no. for resonance must have $\omega = \omega_0$
in this problem $\omega = 2$, $\omega_0 = 3$.

(this solution will be a periodic function. Since we superposed a function with period π with one having period $2\pi/3$, the sum will have a common period ~~g~~ 2π).

4) Consider the sinusoidal function

$$x(t) = 3\cdot\cos(0.2\cdot t) - \sin(0.2\cdot t).$$

4a) Assuming t measures time in seconds, compute the angular frequency, frequency, and period for this function. Use correct units.

(5 points)

$\omega = .2$ rad/sec

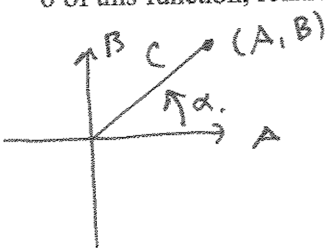
$f = \frac{\omega}{2\pi} = \frac{.2}{2\pi}$ $\left(= \frac{1}{10\pi}\right)$ cycles/sec. $(\approx 0.032$ cycles/sec.$)$

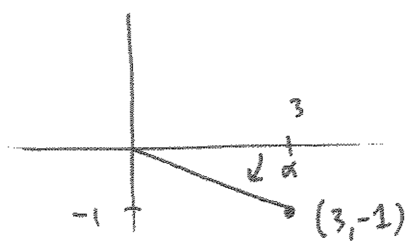
$T = 10\pi$ sec/cycle.

$(\approx 31.4$ sec/cycle$)$

4b) Express $x(t)$ in amplitude-phase form, $x(t) = C\cdot\cos(.2\cdot t - \alpha)$. You may specify α in terms of an inverse trig function, if you don't have a calculator to compute a decimal value. Also specify the time delay δ of this function, relative to $C\cdot\cos(.2\cdot t)$. Use correct units.

(5 points)





$C = \sqrt{9+1} = \sqrt{10} \approx 3.16$ (same units as x, which weren't given)

$\alpha = \arctan\left(-\frac{1}{3}\right)$ $\left(= \arcsin\left(-\frac{1}{\sqrt{10}}\right)\right)$ radians

$\approx -.32$ rad.

(or $\alpha + 2\pi$ works too)

$x(t) = \sqrt{10}\cos(.2t - \alpha)$

$= \sqrt{10}\cos(.2(t - 5\alpha))$

$\delta =$ time delay $= 5\alpha$ sec.

≈ -1.61 sec