Name _____

Student I.D. _____

Section of Math 2250 _____

Math 2250–4
Exam #1
February 17, 2011

   Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right–hand margin. **Good Luck!**

Score
                                POSSIBLE

   1_____ 35

   2_____ 15

   3_____ 15

   4_____ 35

TOTAL_____ 100

1) There has been a pollution accident! A dump truck spilled 50 tons of a water soluable pollutant into Lake Alice. This usually pristine lake has a constant volume of 1000 acre–feet. (An acre–foot is the amount of water in a volume of cross–sectional area one acre, and depth one foot.) Two tributary streams feed pure mountain spring water into Lake Alice at a combined rate of 20 acre–feet per day. A single stream flows out of Lake Alice at a rate of 20 acre–feet per day, maintaining Alice's constant volume. This outlet stream flows directly into Lake Ben, located nearby. Lake Ben has no other tributaries, and its outlet stream also flows at 20 acre–feet per day, keeping Lake Ben's volume at 500 acre–feet.

1a) Use the information above, your differential equations modeling ability, and the assumption that the pollutant in each lake quickly mixes to uniform concentrations, to show that the amount of pollutant in Lake Alice t days after the spill is given by

$$x(t) = 50 \cdot e^{-0.02 \cdot t} \text{ tons}$$

(An input–output diagram might help get you started. The correct differential equation is one of our basic ones, and you can quote the solution rather than rederive it, once you identify the differential equation.)

(10 points)

1b) Using the solution $x(t)$ from part (1a) and the other information in this problem, derive the differential equation governing that the amount of pollutant $y(t)$ in Lake Ben t days after the accident, namely:

$$\frac{dy}{dt} = -0.04 \cdot y(t) + e^{-0.02 \cdot t}.$$

(10 points)

1c) Lake Ben is initially unpolluted. Find a formula for the amount of pollutant $y(t)$ in lake Ben, t days after the spill in upstream Lake Alice. For your reference, here is the differential equation from page 1:

$$\frac{dy}{dt} = -0.04 \cdot y(t) + e^{-0.02 \cdot t}.$$

(10 points)

1d) How many days after the spill is the amount of pollutant in Lake Ben at a maximum? (Your answer may involve logarithms....if you want a decimal approximation to see if your answer is sensible, and you didn't bring a scientific calculator, it may help to know that $\ln(2) = 0.69$, to two decimal places.)

(5 points)

2) Consider a body that moves horizontally through a medium with positive velocity, with drag resistance proportional to the *square* of the velocity and no other forces, so that

$$\frac{dv}{dt} = -k \cdot v^2.$$

For example, this model might describe the deceleration of a boat in water, after the engine is turned off and the boat continues to coast along in a straight line.

2a) Let v_0 be the boat's initial velocity, at time $t = 0$. Derive a formula for the boat's velocity function $v(t)$.

(10 points)

2b) Let x_0 be the boat's initial position. Derive a formula for its position function $x(t)$.

(5 points)

3) Consider the differential equation

$$2 \cdot \frac{dx}{dt} = -x^2 + 4 \cdot x + 5 .$$

3a) Find the equilibrium solutions for this differential equation and use a phase portrait to determine their stability.

(12 points)

3b) If this differential equation was modeling a population, would it be a logistic or doomsday/extinction model? Would it include harvesting or stocking? Briefly justify your answers.

(3 points)

(You are not asked to solve this differential equation, although you should know how to.)

4) Consider the matrix

$$A := \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}.$$

4a) Compute the determinant of A.

(5 points)

4b) What does your computation in 4a tell you about the reduced row echelon form of A? About whether A^{-1} exists? About whether the matrix equation $Ax = b$ has more than one solution x, no solutions x, or unique solutions x? (Here x and b are both column vectors.)

(5 points)

4c) Find A^{-1}, using either of the two algorithms we've discussed in class. Show your work so that it can be checked. (It's probably also wise to check your answer by doing the appropriate matrix multiplication.) For your reference, here again is the matrix A:

(15 points)

$$A := \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}.$$

4d) Let

$$b := \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Solve $Ax = b$ for x, using the inverse matrix you computed in 4c.

(5 points)

4e) Let

$$C := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix}.$$

Solve the matrix equation $XA = C$, for matrix X. (Hint: Use analogous algebraic reasoning to that which justifies the computation you did to solve 4d.)

(5 points)