Math 2250-4
Friday 4/8

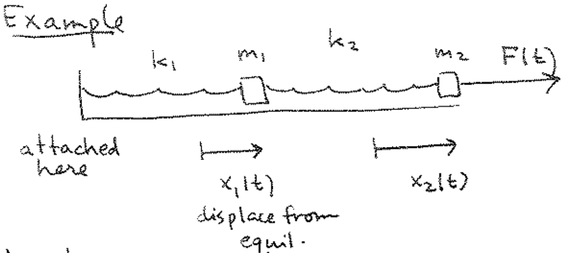
§7.1, hints of 7.2, 7.3.

We saw Tuesday how input-output modeling (compartmental analysis) leads to 1^{st} order systems of DE's. So,

HW for Fri 4/15
7.1 1, 3 5 8 11, (18, 21, 24, 26)
7.2 5, 9, 10 13, (14, 16, 21, 23, 25)
7.3 3, 4, 6 13, (14, 25, 30, 32, 37)
use pplane (same page as dfield) for pictures. In 7.3.6 also solve with Laplace transform.

- What kind of DE's do we get for an interconnected mass-spring system? (or multiply-connected RLC circuits)

Example



Directions for turning in HW this Friday 4/8, or next Friday 4/15, after 12:00 noon: turn HW into

Paula Tooman, front desk of Math office JWB 233, before 5:00 p.m.

NOT TO MY OFFICE

Newton:

$$m_1 x_1''(t) = k_2 \underbrace{(x_2 - x_1)}_{\text{2nd spring is stretched this much}} - k_1 x_1$$

$$m_2 x_2''(t) = -k_2 (x_2 - x_1) + F(t)$$

e.g. $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $f(t) = 40 \sin 3t$ $\Rightarrow$

$$\begin{cases} 2x_1'' = 2(x_2 - x_1) - 4x_1 = -6x_1 + 2x_2 \\ x_2'' = -2(x_2 - x_1) + 40 \sin 3t = 2x_1 - 2x_2 + 40 \sin 3t \end{cases}$$

or $$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \sin 3t \end{bmatrix}$$

- What's a natural IVP for this 2^{nd} order system of 2 linear DE's? (We won't solve it today – see §7.4)

note: Any system of DE's can be converted into an equivalent system of 1^{st} order DE's, by introducing extra unknown functions.

(1) $$\boxed{\begin{aligned} x_1'' &= -3x_1 + x_2 \\ x_2'' &= 2x_1 - 2x_2 + 40 \sin 3t \end{aligned}}$$

Let $y_1 = x_1'$
$y_2 = x_2'$

(2) $$\boxed{\begin{aligned} x_1' &= y_1 \\ y_1' &= -3x_1 + x_2 \\ x_2' &= y_2 \\ y_2' &= 2x_1 - 2x_2 + 40 \sin 3t \end{aligned}}$$

solutions of (1) yield solutions of (2) and vise verse.

This correspondence is sometimes useful.

xample : Study the correspondence between n^{th} order eqtns or systems and 1^{st} order ones : (Every system of arbitrary order DE's can be reduced to an equivalent first order system. The correspondence doesn't always go in the reverse direction)

$x = x(t)$

2nd order linear homog DE

$$x'' - x' - 2x = 0$$

$x = e^{rt}$

$p(r) = r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$$\boxed{x_H(t) = c_1 e^{2t} + c_2 e^{-t}}$$

so we could immediately solve the 1^{st} order system:

$x_H(t) = c_1 e^{2t} + c_2 e^{-t}$

$x_H'(t) = 2c_1 e^{2t} - c_2 e^{-t}$

so $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Also, see work at right!

1st order system of 2 DE's

$y = x'$

$x' = y$

$y' = y + 2x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

try $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \vec{v}$

for $\vec{x}'(t) = A\vec{x}$

$\lambda e^{\lambda t} \vec{v} = A e^{\lambda t} \vec{v}$

$\lambda \vec{v} = A\vec{v}$ $\quad$ $\vec{v}$ eigenvect of A

$0 = (A - \lambda I)\vec{v}$ $\quad$ eigenval λ

$$\det \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = p(\lambda)$$

does this "characteristic polynomial" look familiar?

$= (\lambda - 2)(\lambda + 1)$

$\lambda = 2$

$\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$

$\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenbasis

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

✓

IVP $\begin{cases} x'' - x' - 2x = 0 \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$

$c_1 + c_2 = 2$

$2c_1 - c_2 = 1$

$$\boxed{x(t) = e^{2t} + e^{-t}}$$

IVP $\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$

$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

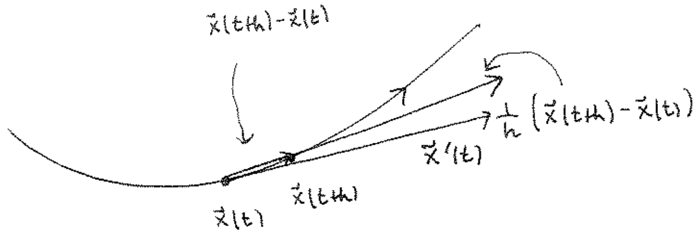
$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Geometric interpretation of 1^{st} order system of DE's:

$$\text{IVP} \begin{cases} \dfrac{d\vec{x}}{dt} = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

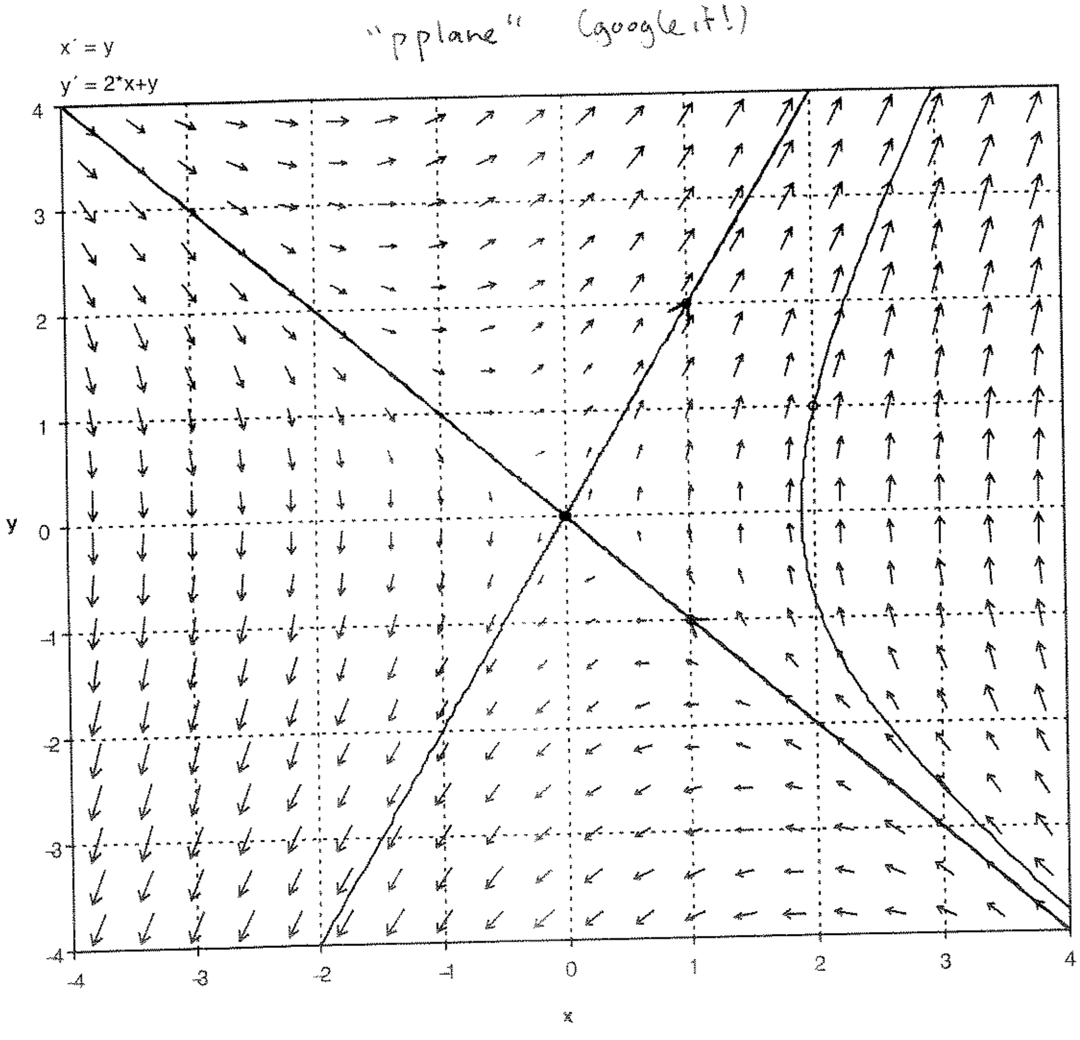
- Recall, if $\vec{x}(t)$ is parametric curve in space (think partical position at time t) then $\vec{x}'(t)$ is the tangent (or velocity) vector:



So the IVP says you know where you start $(\vec{x}(t_0)=\vec{x}_0)$, and you know your velocity vector (depending on time & location) $\longrightarrow$ so you expect a unique sol'n

<u>page 2 example continued</u>:

"pplane" (google it!)



$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\left(c_1 = c_2 = 1 \text{ if } \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

for fun, if we have time:

an example related to §6.3, applications of powers of matrices.

How google search works

I have a more in-depth explanation at
http://www.math.utah.edu/teacherscircle/notes/2_23_2009.pdf

see also wikipedia and
http://www.ams.org/featurecolumn/archive/pagerank.html

A stochastic matrix (p 38[illegible] of our text) is an $n \times n$ matrix with all positive entries, such that the entries in each column add up to 1

e.g. $A = \begin{bmatrix} .1 & .8 & .8 \\ .8 & .1 & .1 \\ .1 & .1 & .1 \end{bmatrix}$

Perron's theorem says any Stochastic matrix has an eigenvalue $\lambda = 1$, $\lambda=1$ eigenspace dim = 1, with an eigenvector $\vec{v}$ with all positive entries. (You can scale $\vec{v}$ so that its entries add up to 1.) All other eigenvalues of a stochastic matrix have absolute value less than 1 (if they're real), or if they're complex, $\lambda = a+bi = \sqrt{a^2+b^2}\left(\underset{\uparrow \atop \cos\theta}{\frac{a}{\sqrt{a^2+b^2}}} + \underset{\uparrow \atop \sin\theta}{\frac{ib}{\sqrt{a^2+b^2}}}\right)$

$= re^{i\theta}$ with $\underline{\underline{r<1}}$

so $\lambda^k = r^k e^{ik\theta}$
$= r^k(\cos k\theta + i \sin k\theta) \to 0$ as $k \to \infty$

thus, for all other λ's,
$\lambda^k \to 0$ as $k \to \infty$.

Thus, If A is stochastic, and is the transition matrix for a discrete dynamical system, then

$\vec{x}_k = A^k \vec{x}_0$.

Express $\vec{x}_0 = \underset{\uparrow \atop \text{the } \lambda=1 \text{ eigenvector}}{c_1\vec{v}_1} + \underbrace{c_2\vec{v}_2 + \dots + c_n\vec{v}_n}$ the eigenvectors (or "generalized" eigenvectors if A is not diagonalizable) with $|\lambda_i| < 1$

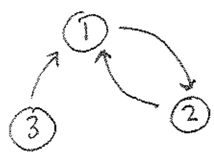
Then $A^k\vec{x}_0 = c_1\vec{v}_1 + c_2 \underbrace{A^k\vec{v}_2}_{\lambda_2^k\vec{v}_2 \to 0} + c_3 A^k\vec{v}_3 + \dots$

$\downarrow \qquad \dots \downarrow$ as $k \to \infty$.
$0 \qquad\quad 0$

So the long-term dynamics are determined by the $\lambda=1$ eigenvector.

The google voting game:

Suppose that for a certain search topic there a 3 sites, and that they link (i.e. with hyperlinks) to each other as follows



Which site should be top ranked?

Initially give each site $1/3$ vote, so $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ is # of votes for sites 1, 2, 3.

to get from stage k to stage $k+1$
each site splits its current votes equally among the sites it links to:
So, in our case

votes for 1 → $x_{k+1} = 0\,x_k + 1\,y_k + 1\,z_k$

" 2 → $y_{k+1} = 1\,x_k + 0\,y_k + 0\,z_k$

" 3 → $z_{k+1} = 0\,x_k + 0\,y_k + 0\,z_k$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}$$

Call this transition matrix B

how site 1 vote is split — how site 2 vote is split — how site 3 vote is split

- column entries always add to 1
- but not all entries are non-zero.

google fudge factor: if there are n sites, and for $\alpha = .15$, make everyone give $\frac{\alpha}{n}$ of their current vote to each site equally, and split the remaining fraction of their vote according to the original recipe.

so $$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = (1-\alpha)\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} .1 & .8 & .8 \\ .8 & .1 & .1 \\ .1 & .1 & .1 \end{bmatrix}$$ (for $\alpha = .3$)

Stochastic!

this is the matrix on page 4

With the google fudge factor, after k iterations

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} .1 & .8 & .8 \\ .8 & .1 & .1 \\ .1 & .1 & .1 \end{bmatrix}^k \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \longrightarrow c_1 \vec{v}_1 \quad \text{as } k \to \infty$$

the $\lambda = 1$ eigenvector

Since total vote is conserved, if $\vec{v}_1$ is the vector with positive entries so that their sum is 1, then since the original vote totalled 1, our answer will actually be $\vec{v}_1$. In fact, we could have apportioned the initial vote however we wanted — as long as the initial vote totalled 1, the limit is $\vec{v}_1$. (So if we initially gave all the vote to site j, which is an initial vote vector

$$\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{entry } j,$$

the limit of $A^k e_j$ is $\vec{v}_1$, which is also the j^{th} column of the limit matrix!

The i^{th} entry of $\vec{v}_1$ is how much vote site i ends up with — the more this number is, the higher the google rank!

```
> with(LinearAlgebra) :
  Digits := 5 :
> A := Matrix(3, 3, [.1, .8, .8, .8, .1, .1, .1, .1, .1]) :
  x := Vector(3, 1, [1/3., 1/3., 1/3.]) :
  map(evalf, A^40.x); # map evalf so that only 5 digits are shown
  map(evalf, A^40);
```

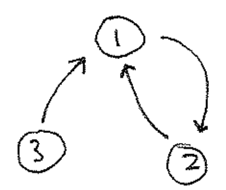
$$\begin{bmatrix} 0.47058 \\ 0.42941 \\ 0.099999 \end{bmatrix}$$

← site 1 is tops

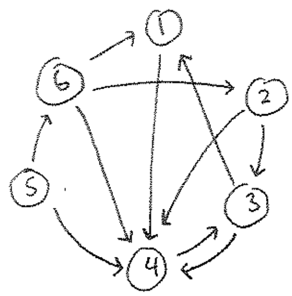
← site 3 is weak

$$\begin{bmatrix} 0.47059 & 0.47059 & 0.47059 \\ 0.42941 & 0.42941 & 0.42941 \\ 0.10000 & 0.10000 & 0.10000 \end{bmatrix}$$

```
>
>
```



A bigger example



$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{site } 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & & & & \end{array} \\
B = & \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}
\end{array}
$$

$\alpha = .15$

$$
B := \begin{bmatrix} 0 & 0 & 0.5 & 0 & 0 & 0.33333 \\ 0 & 0 & 0 & 0 & 0 & 0.33333 \\ 0 & 0.5 & 0 & 1 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 0.5 & 0.33333 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}
$$

$$
goog := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

$$
A := map\left(evalf,\ (1 - \text{alpha}) \cdot B + \frac{\text{alpha}}{n} \cdot goog\right);
$$

$$
A := \begin{bmatrix} 0.025000 & 0.025000 & 0.45000 & 0.025000 & 0.025000 & 0.30833 \\ 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.30833 \\ 0.025000 & 0.45000 & 0.025000 & 0.87500 & 0.025000 & 0.025000 \\ 0.87500 & 0.45000 & 0.45000 & 0.025000 & 0.45000 & 0.30833 \\ 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.025000 \\ 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.45000 & 0.025000 \end{bmatrix}
$$

$> map(evalf, A^{40});$

$$
\begin{bmatrix} 0.18476 & 0.18476 & 0.18476 & 0.18476 & 0.18476 & 0.18476 \\ 0.035093 & 0.035093 & 0.035093 & 0.035093 & 0.035093 & 0.035093 \\ 0.35216 & 0.35216 & 0.35216 & 0.35216 & 0.35216 & 0.35216 \\ 0.36735 & 0.36735 & 0.36735 & 0.36735 & 0.36735 & 0.36734 \\ 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.025000 & 0.024999 \\ 0.035625 & 0.035625 & 0.035625 & 0.035625 & 0.035624 & 0.035624 \end{bmatrix}
$$

← How do you google rank the 6 sites?

$>$

other applications:
any enclosed tank system!
other rankings?
(e.g. BCS football?)

actually, the google guys figured out how to page rank all $\geq 10^7$ sites!