Math 2250-4
Tuesday 4/19

Chapter 9 introduction : non-linear systems of DE's.

e.g.

$$(1)\ \begin{cases} \dfrac{dx}{dt} = F(x,y,t) \\ \dfrac{dy}{dt} = G(x,y,t) \end{cases}$$

system of 2 1st order DE's.

example (9.3):

$x(t)$ = prey population (fish, rabbits, etc.)

$y(t)$ = predator population (sharks, foxes, etc.)

$$\begin{cases} \dfrac{dx}{dt} = ax - pxy \\ \dfrac{dy}{dt} = -by + qxy \end{cases}$$

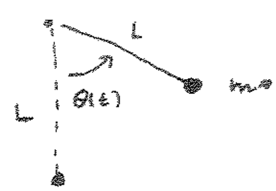
($-cx^2$ if you want logistic prey)

explain model assumptions:

example (9.4):

$x(t)$ = pendulum angle $\Theta(t)$

$y(t) = x'(t)$ = angular velocity



we've derived

$$x'' + \frac{g}{L}\sin x = 0 \text{ , so}$$

$$\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$$

Def: If the only dependence of F & G on t in (1) is through $x(t)$ & $y(t)$ (as in the 2 examples above), the system is called autonomous, i.e.

$$(2)\ \begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

In this case we call the x-y plane the phase plane, and the solution curves $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are called trajectories. They follow the tangent vector field $\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

constant sol'ns to (2) are called <u>equilibrium solutions</u>
They are exactly the sol'ns to the (nonlinear) system

$$(3)\quad \begin{cases} 0 = F(x,y) \\ 0 = G(x,y) \end{cases}$$

<u>example</u> competing species, say $x(t)$ = rabbit population
$y(t)$ = squirrel population

perhaps

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} - \underbrace{xy}_{\text{competition.}}$$

<u>Find the equilibrium sol'ns:</u>

ans (0, 0)
(0, 8)
(7, 0)
(4, 6)

It will be important to know whether equilibrium sol'ns are <u>stable</u> or <u>unstable</u>:

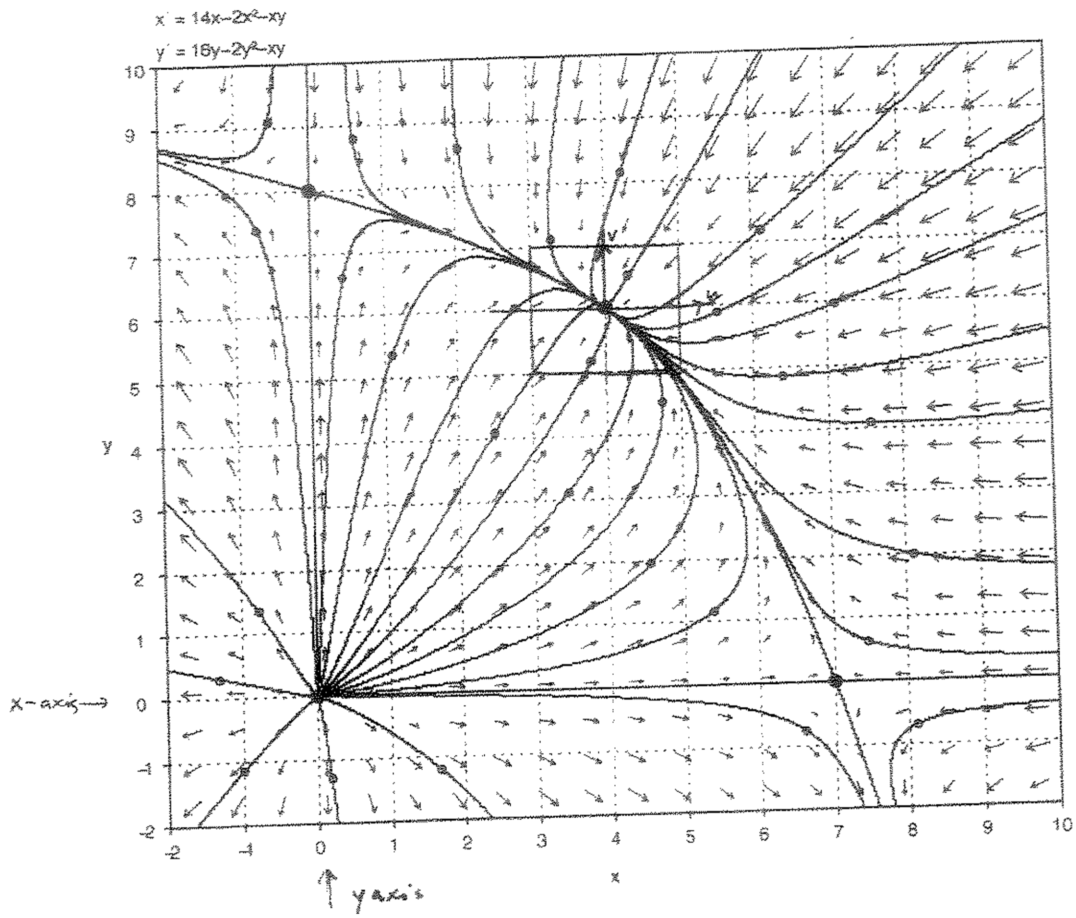
<u>Def</u> $\begin{bmatrix} x_* \\ y_* \end{bmatrix} = \vec{x}^*$ is a <u>stable equilibrium</u> for (2) if it is a const (equil) soltion, i.e. satisfies (3),

<u>and</u> if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.
whenever $\|\vec{x}^* - \vec{x}_0\| < \delta$ $\quad \left(\|\vec{x}^* - \vec{x}_0\| = \sqrt{(x_0 - x_*)^2 + (y_0 - y_*)^2} \right)$
then the sol'n to (2) with $\vec{x}(0) = \vec{x}_0$
satisfies $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon \quad \forall t > 0$.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is <u>unstable equilibrium</u> if it is an equilibrium sol'n that is not stable.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is <u>asymptotically stable</u> iff it's a stable equilibrium, <u>and</u> $\exists \delta > 0$
s.t. $\|\vec{x}^* - \vec{x}_0\| < \delta \Rightarrow$ sol'n to IVP with $\vec{x}(0) = \vec{x}_0$
satisfies $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^*$

Here's the phase portrait (made with pplane) for our competition model:



What happens to rabbit-squirrel populations??

- notice that if we restrict this picture to either x-axis or y-axis (i.e. one pop $\equiv 0$) we get the Chapter 2 1-dim'l phase portraits for logistic DE's.

- discuss apparent stability of the 4 equilibria.

- discuss apparent long term population behavior, assuming both $x_0, y_0 > 0$.

Linearization : Near an equilibrium point, the non-linear system can be effectively approximated with a linear one:

Example : linearize rabbit-squirrel model near $\begin{bmatrix}4\\6\end{bmatrix}$:

let $x = 4+u$ $\quad u=u(t)$ small in abs. val.
$y = 6+v$ $\quad v=v(t)$ small " "

then $\frac{du}{dt} = \frac{dx}{dt} = 14x - 2x^2 - xy = 14(4+u) - 2(4+u)^2 - (4+u)(6+v)$

$= \cancel{56} + 14u \cancel{-32} - 16u - 2u^2 \cancel{-24} - 6u - 4v - uv$

$= -8u - 4v \quad -2u^2 - uv$

$\frac{dv}{dt} = \frac{dy}{dt} = 16y - 2y^2 - xy = 16(6+v) - 2(6+v)^2 - (4+u)(6+v)$

$= \underbrace{96-72-24}_{0} + u(-6) + v(16-24-4) + v^2(-2) + uv(-1)$

$= -6u - 12v \quad -2v^2 - uv$

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \underbrace{\begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}}_{\text{linear piece}} + \begin{bmatrix}-2u^2-uv\\ -2v^2-uv\end{bmatrix}$$

error: if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \delta$ is small, then $\|\text{error}\| \le 3\sqrt{2}\,\delta^2$ is tiny.

in the linearization, discard the error term!

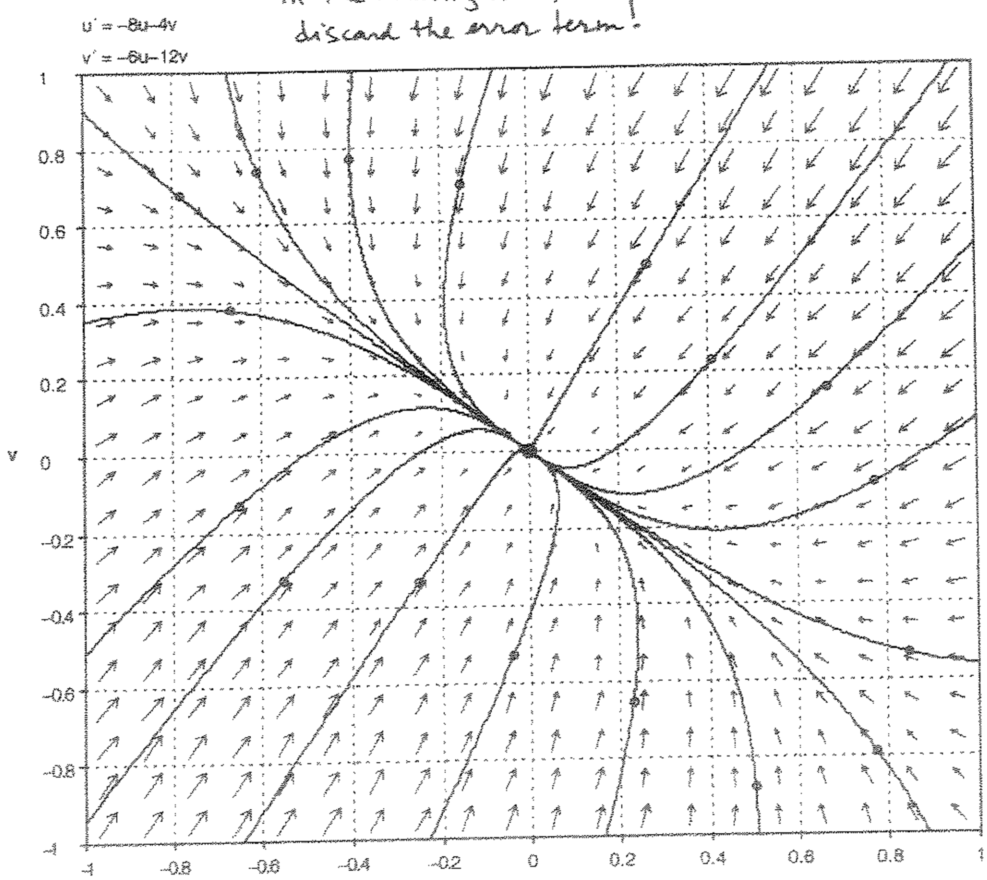
Matrix A has eigendata:

$\lambda = -4.7$, $\lambda = -15.3$

$\vec{v}_1 = \begin{bmatrix}.77\\ -.64\end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix}.49\\ .89\end{bmatrix}$

$\begin{bmatrix}u(t)\\ v(t)\end{bmatrix} \approx c_1 e^{-4.7t}\vec{v}_1 + c_2 e^{-15.3t}\vec{v}_2$

notice how the translation change of variables and linearization yields a picture which essentially just magnifies the little box on page 3!



$\boxed{\text{Shortcut to linearization:}}$ (works for systems of n DE's; illustrated for $n=2$)

Let

$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*,y_*) = F(P) = 0$$
$$G(x_*,y_*) = G(P) = 0$$

write $\quad x(t) = x_* + u(t)$
$\qquad\quad\; y(t) = y_* + v(t)$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\dfrac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v)\to(0,0)$

$$x' = F(x_*+u,\, y_*+v) = F(\cancel{x_*,y_*}) + F_x(x_*,y_*)u + F_y(x_*,y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u,\, y_*+v) = G(\cancel{x_*,y_*}) + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

($F(x_*,y_*) = 0$, $G(x_*,y_*) = 0$)

Math 2210 affine approx.

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of F & G are evaluated at the equil. pt.

$$(2)\qquad \begin{bmatrix} u' \\ v' \end{bmatrix} \cong \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"A"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in A are evaluated at (x^*,y^*))

this is the linearization of (1), at (x^*,y^*). (We use the same letters u, v as we did for the non-linear problem, even though the sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the linearized problem only approximates the translated sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the non-linear problem)

the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

the matrix A is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

<u>Exercise</u>

Check your page 4 work with the Jacobian shortcut!