Math 2250-4
Monday April 18

§7.5 Defective eigenspaces.

and how to find solution space to

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

when A is <u>not</u> diagonalizable.

<u>Example</u> Find the solution space to

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \vec{x}$$

<u>step 1</u> (as always).

$$\begin{vmatrix} 3-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = (\lambda-3)^2$$

$\lambda = 3$:

$$\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 0 & 0 \end{array}$$

eigenbasis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$v_2 = 0$

$\vec{v} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

<u>one</u> sol'n $e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; need another independent sol'n!

<u>discussion</u> : Let $A\vec{v} = \lambda_i \vec{v}$, but the λ_i-eigenspace is "defective" i.e. $\dim(\lambda_i\text{-eigenspace}) < k_i$ where charact poly has factor $(\lambda-\lambda_i)^{k_i}$.

Does multiplying by t work?

i.e. we know (for $\lambda = \lambda_i$)

$\vec{x}(t) = e^{\lambda t}\vec{v}$ solves $\vec{x}' = A\vec{x}$

does $\vec{z}(t) = t e^{\lambda t}\vec{v}$:

$$\vec{z}' = \left(e^{\lambda t} + t\lambda e^{\lambda t}\right)\vec{v} \qquad A\vec{z} = te^{\lambda t}A\vec{v} = t\lambda e^{\lambda t}\vec{v}$$

failure.

well, how about

$$\vec{z}(t) = e^{\lambda t}(t\vec{v} + \vec{w}) \quad (\text{same } \vec{v})$$

$$\vec{z}'(t) = \left[e^{\lambda t} + t\lambda e^{\lambda t}\right]\vec{v} + \lambda e^{\lambda t}\vec{w}$$

$$A\vec{z} = t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}A\vec{w}$$

Success if

$$A\vec{w} = \lambda\vec{w} + \vec{v}$$

"a chain of length 2"

$$\boxed{\begin{aligned} (A-\lambda I)\vec{w} &= \vec{v} \\ (A-\lambda I)\vec{v} &= 0 \end{aligned}}$$

last page

<u>in our example</u>

$(A-\lambda I)\vec{w} = \vec{v}$:

$$\begin{array}{cc|c} 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}$$

$4w_2 = 1$ e.g. $\vec{w} = \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}$

$$\vec{z}(t) = e^{3t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}\right)$$

General sol'n:

$$\vec{x}_H(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}\right)$$

Theorems: Let $A_{n\times n}$ as on page 1, with

$$|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m}$$

$k_1+\dots+k_m=n$
$\lambda_i \in \mathbb{C} \quad 1\le i\le m$
λ_i distinct

(1) The generalized λ_i-eigenspace is defined to be

$$\ker (A-\lambda_i I)^{k_i}$$

it is always k_i dimensional (in $\mathbb{C}^n$)

(2) By amalgamating bases of the generalized eigenspaces you get a basis for $\mathbb{C}^n$

(3) The generalized λ_i-eigenbases can be chosen to be made of one or more "chains" $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ where for $\lambda=\lambda_i$

$$\begin{aligned}(A-\lambda I)\vec{u}_1 &= \vec{0} \\ (A-\lambda I)\vec{u}_2 &= \vec{u}_1 \\ &\vdots \\ (A-\lambda I)\vec{u}_k &= \vec{u}_{k-1}\end{aligned}$$

($\vec{u}_1$ is an eigenvector)

(on page 1, $(\vec{v},\vec{w})$ was a chain of length 2)

(4) Each chain $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ yields k lin. ind. sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$:

$$\begin{aligned}\vec{x}_1(t) &= \vec{u}_1 e^{\lambda t} \\ \vec{x}_2(t) &= (\vec{u}_1 t + \vec{u}_2) e^{\lambda t} \\ \vec{x}_3(t) &= (\vec{u}_1 \tfrac{1}{2} t^2 + \vec{u}_2 t + \vec{u}_3) e^{\lambda t} \\ &\vdots \\ \vec{x}_k(t) &= (\vec{u}_1 \tfrac{1}{k!} t^k + \dots + \vec{u}_{k-1} t + \vec{u}_k) e^{\lambda t}\end{aligned}$$

$\leftarrow$ notice $\vec{x}_j{}' = \lambda \vec{x}_j + \vec{x}_{j-1}$
$A\vec{x}_j = \lambda\vec{x}_j + \vec{x}_{j-1}$ too!

amalgamating these solutions yields a basis of sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$!!!

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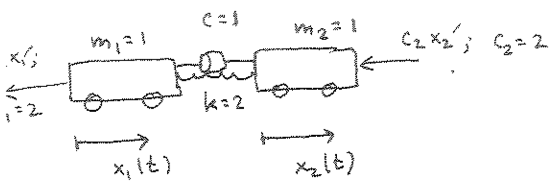


$$m_1x_1'' = +k(x_2-x_1) - c_1x_1' + c(x_2'-x_1')$$
$$m_2x_2'' = -k(x_2-x_1) - c_2x_2' - c(x_2'-x_1')$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

cannot use § 5.3 (no damping.)
Must convert to 1st order system

$x_1 = x_1$
$x_2 = x_2$
$x_3 = x_1'$
$x_4 = x_2'$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Maple to the rescue! (See next page).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t}\left[c_2\begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + c_4\underbrace{\left[t\begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right]}_{t\vec{u}_1 + \vec{u}_2} \right]$$

explain.

nice physically; the sum of MAPLE's 2 $\lambda=-2$ eigenvects, times -2.

the difference of Maple's 2 $\lambda=-2$ eigenvects, ×2. also nice physical interpretation defective eigenspace computations

```
> with(LinearAlgebra) :
> A := Matrix(4, 4, [0, 0, 1, 0, 0, 0, 0, 1,
            -2, 2,-3, 1, 2,-2, 1,-3]);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \quad (1)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 0 \\ -2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2)$$

```
> Iden := Matrix(4, shape = identity);
```

$$Iden := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

> $B := A + 2 \cdot Iden$; *#eigenvalue of A is −2, this is* $A - \lambda \cdot I$

$$B := \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \quad (4)$$

> $NullSpace(B^3)$; *# the nullspace of* $(A-\lambda \cdot I)^k$ *is k − dimensional, where k is the algebraic multiplicity of* λ
$NullSpace(B^2)$;
$NullSpace(B)$;

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \quad (5)$$

>

From the above we deduce that the chain structure of the generalized $\lambda = -2$ nullspace is one chain of length two and one chain of length 1.

```
> u2 := Vector([0, 0,-1, 1]);
      #for chain of llength 2 don't start with an eigenvector.  I like this one for its physical meaning -
    #it's the sum of the first two generalized eigenvectors.
  u1 := B.u2;
```

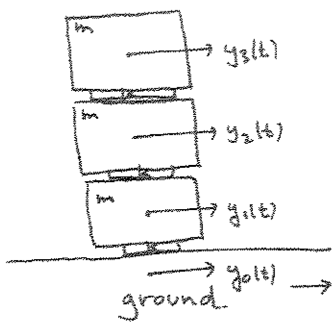
$$u2 := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$u1 := \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad (6)$$

Earthquake Project Comments

(A careful explanation of the book's claim that even though the building is only being shaken at the base, in a frame of reference moving with ground, each floor experiences an equal "inertial force".)

consider a 3 story building (in project is 7 stories) modeled as a mass-spring model, with each story having mass m, and floor junctions modeled as "springs" with constants k (with respect to transverse as opposed to parallel motion)



$\rightarrow$ transverse (shaking) wave $E\cos\omega t$

$y_0(t) = E\cos\omega t$ so

$$y_0'' = -E\omega^2\cos\omega t$$
$$m y_1'' = -k(y_1-y_0) + k(y_1-y_2)$$
$$m y_2'' = -k(y_2-y_1) - k(y_2-y_3)$$
$$m y_3'' = -k(y_3-y_2)$$

reduction of fns

let

$$x_0 = y_0 - E\cos\omega t \equiv 0$$
$$x_1 = y_1 - E\cos\omega t$$
$$x_2 = y_2 - E\cos\omega t$$
$$x_3 = y_3 - E\cos\omega t$$

$$x_0 \equiv 0$$
$$m x_1'' = m y_1'' + mE\omega^2\cos\omega t$$
$$m x_2'' = m y_2'' + mE\omega^2\cos\omega t$$
$$m x_3'' = m y_3'' + mE\omega^2\cos\omega t$$

so

$$m x_1'' = -k(x_1-0) - k(x_1-x_2) + mE\omega^2\cos\omega t$$
$$m x_2'' = -k(x_2-x_1) - k(x_2-x_3) + mE\omega^2\cos\omega t$$
$$m x_3'' = -k(x_3-x_2) + mE\omega^2\cos\omega t$$

so the x's are the displacements from equilibrium as measured by an observer on the ground

Notice $y_i - y_j = x_i - x_j$

$$\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} -2k/m & k/m & 0 \\ k/m & -2k/m & k/m \\ 0 & k/m & -k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + E\omega^2\cos\omega t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

it's as if all floors are being forced - book calls this inertial forcing due to frame of reference