Math 2250-4
Fri. Apr. 15

HW for Fri. Apr. 22: §7.4 earthquake proj — see HW page for details.

also, 7.4 2, 3, 8, 13, 14, 16, 17, 18
9.1 5, 8, 11, 15, 20, 24
9.2 5, 6, 7, 8, 9, 14, 15, 19, 27, 30.

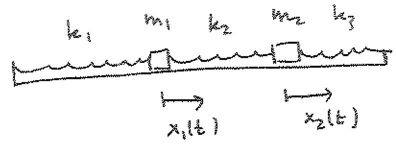
use pplane to draw phase portraits — google it!

§7.4b: forced undamped mass-spring systems

Last lecture we discussed unforced undamped spring systems:

$$M\vec{x}'' = K\vec{x}$$

e.g. (done Wed)



$$m_1 x_1'' = k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 x_2'' = -k_3 x_2 - k_2(x_2 - x_1)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2-k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M\vec{x}'' = K\vec{x}$$

which (multiplying by M^{-1} on the left) yields

$$\vec{x}'' = A\vec{x} \qquad (A = M^{-1}K)$$

"A" for acceleration!

e.g.
$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2-k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Today let's force the same mass-spring systems!
(and study practical resonance for related slightly damped systems).

$$M\vec{x}'' = K\vec{x} + \vec{F}(t)$$

$$\boxed{* \qquad \vec{x}'' = A\vec{x} + \vec{f}(t)}$$

$A = M^{-1}K$
$\vec{f} = M^{-1}\vec{F}$

we study the case $\boxed{\vec{f}(t) = \vec{F}_0 \cos\omega t}$, so we're forcing each mass with the same angular frequency ω, but with different amplitudes for each mass.
(In practice all of the amplitudes except one may be zero).

$\vec{F}_0$: constant vector of ± forcing amplitudes on each mass

General sol'n to $*$ (linear nonhomogeneous sys of DE's) is

$$\vec{x} = \vec{x}_p + \vec{x}_H$$

$\vec{x}_H$: discussed Wed: basis sol'ns of form $\cos\omega_i t\,\vec{v}$, $\sin\omega_i t\,\vec{v}$ with $A\vec{v} = -\omega_i^2\vec{v}$.

try $\vec{x}_p = \vec{c}\cos\omega t$ (method of undetermined coeff's)

$*$ says $-\omega^2 \cancel{\cos\omega t}\,\vec{c} = \cancel{\cos\omega t}\,A\vec{c} + \cancel{\cos\omega t}\,\vec{F}_0$

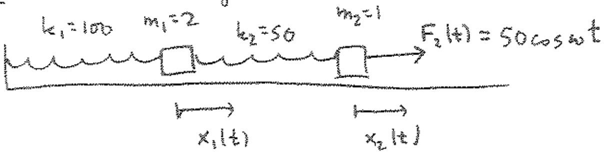
$$-\vec{F}_0 = A\vec{c} + \omega^2\vec{c} = (A + \omega^2 I)\vec{c}.$$

$\boxed{-\vec{F}_0 = (A+\omega^2 I)\vec{c}}$

$$\boxed{\vec{c} = -(A + \omega^2 I)^{-1}\vec{F}_0}$$

works for $\omega \neq \omega_i$, one of the natural frequencies

Example For the system



A) Using Newton's law (i.e. don't just plug into page 1) derive the 2^{nd} order system of DE's that governs this system, in the form of $*$ on page 1

Ans

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos\omega t \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

B) Find $\vec{x}_H$ for this system, using the algorithm we discussed on Wed.
(Example 1 page 435)

Ans

$$\vec{x}(t) = \underbrace{\underbrace{(c_1\cos 5t + c_2\sin 5t)}_{C_1\cos(5t-\alpha)} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{(slower) in phase fundamental mode}} + \underbrace{\underbrace{(c_3\cos 10t + c_4\sin 10t)}_{C_2\cos(10t-\alpha)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{(faster) opposite phase fund. mode}}$$

C) Find $\vec{x}_p = \cos\omega t\, \vec{c}$ to the inhomog. system

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos\omega t \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

(Example 3 p. 441)
(Adjoint inverse formula or Cramer's rule will help here.)

ans

$$\vec{x}_p(t) = \cos\omega t \begin{bmatrix} \dfrac{1250}{(\omega^2-25)(\omega^2-100)} \\ \dfrac{50(\omega^2-75)}{(\omega^2-25)(\omega^2-100)} \end{bmatrix}$$

resonance

D) <u>Interpretation</u>: Notice our $\vec{x}_p$ doesn't work for $\omega = 5$ or $\omega = 10$. (because we'd get resonance instead!)

beating

Even when $\omega \neq 5, 10$, if ω is near 5 or 10 the general soln

$$\vec{x}(t) = \vec{x}_p + \vec{x}_H$$

$\vec{x}_p$: ang freq ω — $\vec{x}_H$: superposition with angular frequencies 5, 10

is likely to exhibit large amplitude beating behavior because of the interaction between ω and its nearby natural angular frequency.

practical resonance

In any "real" demonstration of this model there will be some damping. Then, for the model which takes damping into account,

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$$

$\vec{x}_p(t)$: $\cos\omega t\, \vec{c}(\omega) = \vec{x}_{sp}(t)$ (steady periodic)

$\vec{x}_H(t)$: $x_{tr}(t)$ (transient because of damping)

will be <u>close</u> to undamped soltn, ($\alpha \approx 0$, $\vec{c}(\omega)$ almost the same) as computed in example above. Thus we can study practical resonance by computing how large the entries of $\vec{c}$ are, depending on ω.

MATH 2250-4
Forced Mass–Spring systems
April 15, 2011

This handout covers section 7.4 and might help you for the Earthquake exploration you're doing at the end of that section, see pages 444–445 and our homework page. We're working this example out by hand in class as well.

```
> with(LinearAlgebra): with(plots): with(DEtools):
> M:=Matrix([[2,0],[0,1]]);
  K:=Matrix([[-150,50],[50,-50]]);
  A:=M^(-1).K   ;#period to multiply
```

$$M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K := \begin{bmatrix} -150 & 50 \\ 50 & -50 \end{bmatrix}$$

$$A := \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \tag{1}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -100 \\ -25 \end{bmatrix}, \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \tag{2}$$

Therefore, the natural frequencies of this system are the 10 and 5, and the two fundamental modes correspond to the masses moving in opposite directions (with equal amplitudes and angular frequency 10) and in parallel directions (with amplitude ratio of two and angular frequency 5), as we work out by hand today in our handwritten class notes.

Now, let's consider the forced system with force vector equal to $\cos(\omega \cdot t)$ [50, 0], i.e. the second mass is being forced periodically. In other words, the system **Mx''=Kx + F**, where F=cos(wt)[0,50] ; this is discussed in general on page 440, and in this particular instance as Example 3 on page 441. These computations parallel our hand computations.

```
> F0 := M^(-1).Vector([0, 50]) :
        #The F0 in the normalized equation (30), page 440...
        # needs serious modification for Maple project
  Iden := Matrix(2, shape = identity) :
        # the 2 by 2 identity matrix
  Aleft := ω→A + ω^2·Iden :
      # the matrix function multiplying
      # c on the left side of (32)
  c := ω→(Aleft(ω))^(-1).(-F0) :
      # the solution vector c(omega) to (32),
      # obtained by multiplying both sides of equation
      # (32) on the left, by the inverse to Aleft

> c(ω);  #should agree with equation (35) page 441
     #you won't want to display c(ω) in the Maple project
     #because it will take multiple pages.
```

$$\begin{bmatrix} \dfrac{1250}{2500 - 125\,\omega^2 + \omega^4} \\[2ex] -\dfrac{50\left(-75 + \omega^2\right)}{2500 - 125\,\omega^2 + \omega^4} \end{bmatrix} \qquad \textbf{(3)}$$

The vector $c(\omega)$, as above, times the function of time, $\cos(\omega \cdot t)$, is a particular solution to the forced oscillation problem we are considering. If we assume that our actual problem has a small amount of damping, then we expect that this particular solution is very close to the steady periodic solution to the damped problem. Read the discussion on page 441. We can study practical resonaonace phenomena for these slightly damped problmes by plotting the maximum amplitude for the individual mass displacements, for these particular steady state solutions to the undamped problems. Use the Maple command ''norm'' to measure this maximum amplitude of the individual mass oscillations:

```
> norm(c(omega));
```

$$\max\left(\frac{1250}{\left|2500-125\,\omega^2+\omega^4\right|}, 50\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|\right) \tag{4}$$

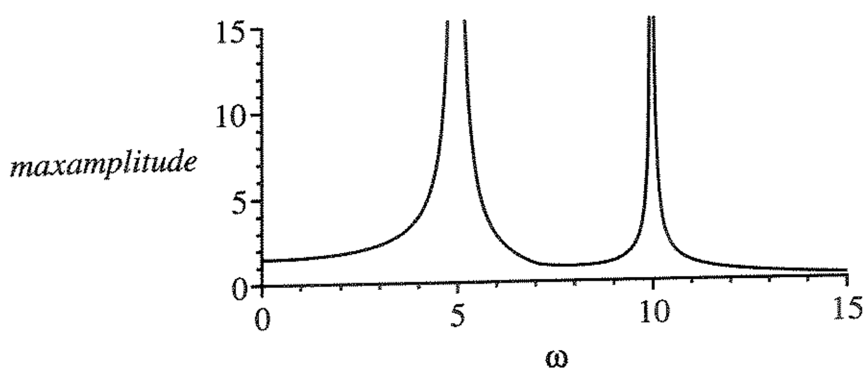
Another way to measure the size of $c(\omega)$ is to take the Euclidean magnitude, which is the command

```
> norm(c(ω), 2);
```

$$50\sqrt{\frac{625}{\left|2500-125\,\omega^2+\omega^4\right|^2}+\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|^2} \tag{5}$$

(You will use the first command in your Maple project, which perhaps makes the most sense since it will be measuring the maximum amplitude that any floor oscillates relative to the ground.) The following picture illustrates that the maximum amplitutde of the particular solution blows up when ω is near the two natural angular frequencies. Thus, in the slightly damped problem, one would experience practical resonance in the steady periodic solution (which would be very close to the steady periodic solution in our undamped problem).

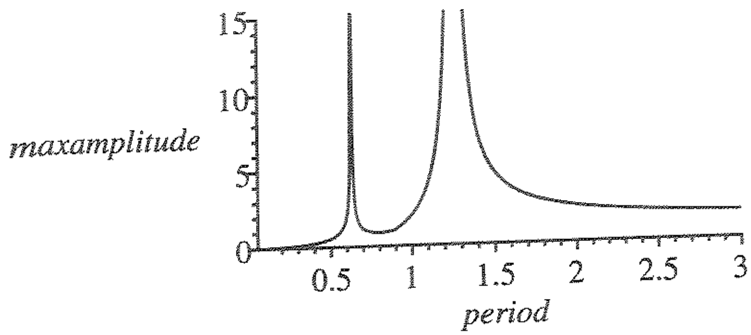
```
> plot(norm(c(omega)),omega=0..15,maxamplitude=0..15,
        numpoints=200,color='black');
```



This is qualitatively the picture on page 441, figure 7.4.10, although they plotted the (Euclidean) magnitude of c(omega) rather than the maximum of the individual amplitudes. Notice how we get Maple to label the axes as desired.

We can get a plot of resonance as a function of period by recalling that $\frac{2 \cdot \pi}{T} = \omega$:

```
> plot(norm(c(2*Pi/period)),period=0.1..3,maxamplitude=0..15,
  numpoints=200,color='black');
```




```
>
```