

We've been discussing first order systems of differential equations, and the natural initial value problem

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

(and that any system of higher order differential equations can be studied in terms a related system of 1st order DE's.)

- A good place to re-enforce this discussion is page 2: the undamped, unforced harmonic oscillator
- Then: re-interpreted as a system of 1st order DE's.

General result

$$\text{IVP} \quad \begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

If $\vec{f}$ is differentiable in its variables the there exists a unique sol'n to IVP, defined on some interval $(t_0 - \delta, t_0 + \delta)$

Linear systems of DE's result

$$\text{IVP} \quad \begin{cases} \frac{d\vec{x}}{dt} = P(t) \vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

- $P(t)$ is an $n \times n$ matrix fn of t
- system is homogeneous if $\vec{f}(t) \equiv 0$
(else inhomogeneous)

If $P(t)$ and $\vec{f}(t)$ are continuous on an interval containing t_0 , then there exists a unique sol'n to IVP, defined on the entire interval.

The harmonic oscillator
(unforced, undamped, $\omega_0 = 1$)

(2)

$$x''(t) + x(t) = 0$$

$$\omega_0 = 1.$$

$$x(t) = C \cos(t - \alpha).$$

DE $\rightarrow$ 1st order system

so solns to system are

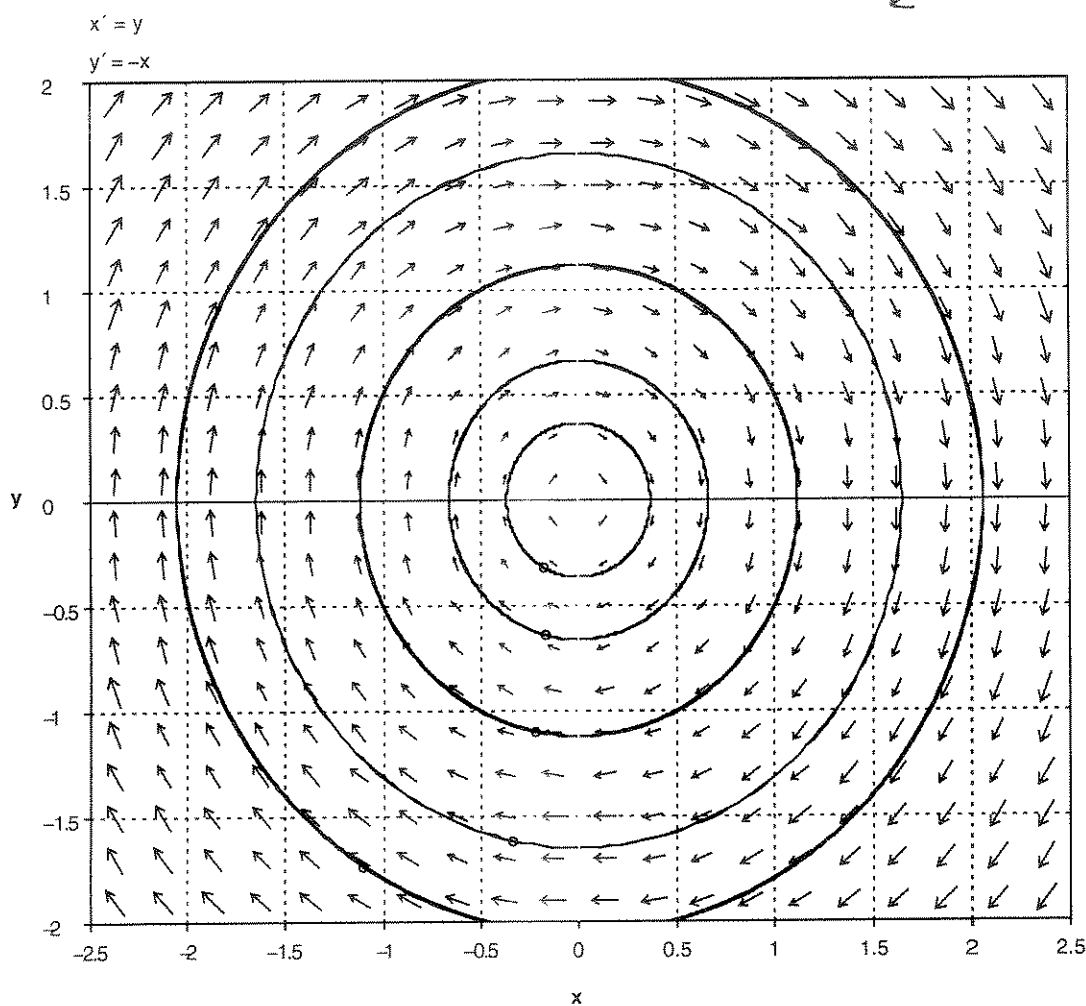
$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C \cos(t - \alpha) \\ -C \sin(t - \alpha) \end{bmatrix}$$

$$= C \begin{bmatrix} \cos(t - \alpha) \\ -\sin(t - \alpha) \end{bmatrix}$$

clockwise circles in phase portrait!

$$\begin{aligned} x' &= y \\ \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} y \\ -x \end{bmatrix} \leftarrow \text{by def.} \\ &\leftarrow \text{from original DE.} \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Exercise: Find a basis of solns to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ using eigenvalue-eigenvector method!

Systems of linear 1st order DE's theory (this should feel familiar)

① Homogeneous system

$$* \frac{d\vec{x}}{dt} = P(t)\vec{x}$$

- solution space is a subspace of $\vec{f}$: vector-valued fns of t

Let $\vec{x}(t)$ and $\vec{z}(t)$ solve *

Then

$$\frac{d}{dt} (c_1 \vec{x}(t) + c_2 \vec{z}(t)) = c_1 \frac{d\vec{x}}{dt} + c_2 \frac{d\vec{z}}{dt}$$

$$= c_1 (P(t)\vec{x}) + c_2 (P(t)\vec{z})$$

because $\vec{x}(t)$,
 $\vec{z}(t)$ are sol'n

$$= P(t) (c_1 \vec{x}(t) + c_2 \vec{z}(t))$$

Matrix properties

so linear combos of sol'n's

are sol'n's $\rightarrow$ this condition (closure under + & scalar mult)
is the subspace condition.

- the solution subspace to *

is n dimensional (if $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$)

Because, if

$$\vec{x}_j(t) \text{ solves } \begin{cases} \frac{d\vec{x}}{dt} = P(t)\vec{x} \\ \vec{x}_j(t_0) = \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \leftarrow \text{entry } j \end{cases}$$

then $\{\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)\}$

are a basis:

$$\text{span: The sol'n to } \begin{cases} \frac{d\vec{x}}{dt} = P(t)\vec{x} \\ \vec{x}(t_0) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{cases}$$

$$\text{is } \vec{x}(t) = b_1 \vec{x}_1(t) + b_2 \vec{x}_2(t) + \dots + b_n \vec{x}_n(t)$$

independent: if $c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \dots + c_n \vec{x}_n(t) \equiv \vec{0}$

then at t_0 : $c_1 \vec{e}_1 + c_2 \vec{e}_2 + \dots + c_n \vec{e}_n = \vec{0}$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



2) Inhomogeneous 1st-order systems of DE's.

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

The general solution is

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$$

where $\vec{x}_p(t)$ is a particular sol'n and $\vec{x}_H(t)$ is the general homogeneous sol'n

proof $\mathcal{L}(\vec{x}) := \frac{d\vec{x}}{dt} - P(t)\vec{x}$ is a linear operator so the proof is virtually identical to the discussion for nonhomogeneous linear differential equations of order n .

Example

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$$

$$\vec{x}_H(t): \quad \begin{array}{ccc} & \uparrow & \uparrow \\ & P(t) & \vec{f}(t) \end{array}$$

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> with(LinearAlgebra):  
A := Matrix(3, 3, [3, -2, 0, -1, 3, -2, 0, -1, 3]);  
Eigenvectors(A);
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$$A := \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$$

eigenvectors

$$\begin{array}{l} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\ \text{eigenvalues} \rightarrow \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \end{array}$$

Maple tells us

$$\vec{x}_H(t) = c_1 e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

← Note "Wronskian" $W(t) = \left| e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right|$

(if our system came by converting a DE, this is the old Wronskian!)

$$= e^{9t} \begin{vmatrix} 2 & -2 & 2 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = e^{9t} (-32) \neq 0$$

So we have all sol's,

i.e.

$$\left\{ e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

are a basis (they're independent and span because we can solve any IVP with them.)

so our 3 vector-valued sol's are linearly independent

(5)

$\vec{x}_p(t)$: Try $\vec{x}_p(t) = \vec{c}$ (a constant vector)

then $\frac{d\vec{x}_p}{dt} = \frac{d}{dt} \vec{c} = 0 \stackrel{?}{=} A\vec{c} + \begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$

$$\vec{c} = -A^{-1} \begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

> $v := \text{Vector}([5, -15, 10]);$
 $c := -A^{-1}.v;$

$$v := \begin{bmatrix} 5 \\ -15 \\ 10 \end{bmatrix}$$

$$c := \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Full solution

$$\vec{x}(t) = \vec{x}_p + \vec{x}_h$$

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + c_1 e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$