Math 2250-004
Week 8: Review and finish sections 4.2-4.4 and linear combination concepts, and then begin Chapter 5 on linear differential equations, sections 5.1-5.2.

Monday Oct 17
This week's notes incorporate some of the material from the week 7 notes that we did not get to. Recall the following concepts from before break:

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

any vector $\underline{\boldsymbol{v}}$ *that is a sum of scalar multiples of those vectors, i.e. any* $\underline{\boldsymbol{v}}$ expressible as

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n.$$

The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is

the collection of all possible linear combinations:

$$span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\} := \{\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n \text{ such that each } c_i \in \mathbb{R}, i = 1, 2, \ldots n\}$$

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent means

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = 0 \Rightarrow c_1 = c_2 = \ldots = c_n = 0.$$

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent means

they are not linearly independent.

A subset W of $\mathbb{R}^n$ is a subspace means

that W is closed under addition and scalar multiplication:

$$(\alpha) \qquad \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in W \Rightarrow \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in W,$$
$$(\beta) \qquad \underline{\boldsymbol{u}} \in W, c \in \mathbb{R} \Rightarrow c\,\underline{\boldsymbol{u}} \in W.$$

Let W be a subspace. A basis for W is

a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ *that are linearly independent and span W.*

Equivalently, each $\underline{\boldsymbol{w}} \in W$ can be written as $\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$ *for exactly one choice of linear combination coefficients* $c_1, c_2, \ldots c_n$.

The dimension of a subspace W is the number of vectors in a basis for W. (It turns out that all bases for a subspace always have the same number of vectors.)

Note: Subspaces are special subsets because they are closed with respect to all linear combinations (since linear combinations are built up by successive scalar multiplication and addition operations). The reason why we use the word "subspaces" for these special subsets is because there is a more general notion of vector space for collections of objects that can be added and scalar multiplied so that the usual addition and scalar multiplication axioms hold. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, .. $\mathbb{R}^m$ are vector spaces. Subspaces of $\mathbb{R}^m$ are also vector spaces in their own right. In Chapter 5 we'll be focusing on function vector spaces (since you can add and scalar multiply functions), and subspaces of those. The purpose will be to understand the solution spaces for higher order linear differential equations, and applications.

Key facts about subspaces:

There are two ways that subspaces arise: We discussed the first way before break. The second way is also important. These ideas will be important when we return to differential equations, in Chapter 5.

1) $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$.

Expressing a subspace this way is an explicit way to describe the subspace W, because you are "listing" all of the vectors in it. In this case we prefer that $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ be linearly independent, i.e. a basis, because that guarantees that each $\underline{\boldsymbol{w}} \in W$ is a unique linear combination of these spanning vectors.

Recall why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$$

$$\underline{\boldsymbol{w}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$$

$$\Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} = (c_1 + d_1)\underline{\boldsymbol{v}}_1 + (c_2 + d_2)\underline{\boldsymbol{v}}_2 + \ldots + (c_n + d_n)\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \alpha)$$

and let $c \in \mathbb{R} \Rightarrow$

$$c\underline{\boldsymbol{v}} = cc_1\underline{\boldsymbol{v}}_1 + cc_2\underline{\boldsymbol{v}}_2 + \ldots + cc_n\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \beta)$$

2) $W = \{\underline{\boldsymbol{x}} \in \mathbb{R}^n \textit{ such that } A_{m \times n}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}\}$.

This is an implicit way to describe the subspace W because you're only specifying a homogeneous matrix equation that the vectors in W must satisfy, but you're not saying what the vectors are.

Why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}},\ A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A\underline{\boldsymbol{v}} + A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A(\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}}) = \underline{\boldsymbol{0}} \Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} \in W \quad \text{(verifies } \alpha)$$

and let $c \in \mathbb{R} \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \Rightarrow cA\underline{\boldsymbol{v}} = c\underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} \Rightarrow A(c\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}} \Rightarrow c\underline{\boldsymbol{v}} \in W \quad \text{(verfies } \beta).$$

....

Example: In the quiz before break you were given an explicit description of a subspace of $\mathbb{R}^3$ as a span of two linearly independent vectors:

$$W = span\left\{\underline{\boldsymbol{u}} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},\ \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}\right\}.$$

You found an implicit description of this 2-dimensional plane by figuring out which points $[x, y, z]^T$ were in the span. This implicit description was actually as the solution space to a homogeneous matrix equation:

$$\begin{bmatrix} 2 & 1 & x \\ 3 & -1 & y \\ 1 & 3 & z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & z \\ 0 & 5 & -x + 2z \\ 0 & 0 & -2x + y + z \end{bmatrix}$$

$$\Rightarrow -2x + y + z = 0$$

$$[-2, 1, 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Exercise 1) Use properties of reduced row echelon form matrices to answer the following questions:
1a) Why must more than two vectors in $\mathbb{R}^2$ always be linearly dependent?
1b) Why can fewer than two vectors (i.e. one vector) not span $\mathbb{R}^2$?
(We deduce from 1a,b that every basis of $\mathbb{R}^2$ must have exactly two vectors.)
1c) If $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ are any two vectors in $\mathbb{R}^2$ what is the condition on the reduced row echelon form of the 2×2 matrix $\langle \underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^2$? That guarantees they're a basis for $\mathbb{R}^2$?
1d) What is the dimension of $\mathbb{R}^2$?

Exercise 2) Use properties of reduced row echelon form matrices to answer the following questions:
2a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?
2b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)
2c) If you are given 3 vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3×3 matrix $\langle \underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \underline{\boldsymbol{v}}_3 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?
2d) What is the dimension of $\mathbb{R}^3$?

Exercise 3) Most subsets of a vector space $\mathbb{R}^m$ are actually not subspaces. Show that

3a) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ x^2 + y^2 = 4 \}$ is not a subspace of $\mathbb{R}^2$.

3b) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ y = 3\,x + 1 \}$ is not a subspace of $\mathbb{R}^2$.

3c) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ y = 3\,x \}$ is a subspace of $\mathbb{R}^2$. Then find a basis for this subspace.

3d) W = the solution space in $\mathbb{R}^n$ to a non-homogeneous matrix equation $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ ($\underline{\boldsymbol{b}} \neq \underline{\mathbf{0}}$), with $A_{m \times n}$ is not a subspace. In particular, lines and planes that don't go through the origin are not subspaces.

Exercise 4) Use geometric reasoning to argue why the only subspaces of $\mathbb{R}^2$ are
(0) The single vector $[0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) All of $\mathbb{R}^2$.

Exercise 5) Use matrix theory to show that the only subspaces of $\mathbb{R}^3$ are
(0) The single vector $[0, 0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) A plane through the origin, i.e. $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ where $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ are linearly independent, or
(3) All of $\mathbb{R}^3$.

Exercise 6) What are the dimensions of the subspaces in Exercise 4 and Exercise 5? How do these ideas generalize to subspaces of $\mathbb{R}^n$?

Usually in applications we do not start with a basis for a subspace - rather this is the goal we search for, since the entire subspace may be reconstructed explicitly and precisely from the basis (which is why a basis is called "a basis"). Usually, our subspace W in $\mathbb{R}^m$ is likely to be described in an implicit manner, as the solution space to a homogeneous matrix equation. In Chapter 5 the subspaces W will be the solution spaces to "homogeneous linear differential equations."

<u>Exercise 7)</u> Use Chapter 3 techniques to find a basis for the solution space $W \subseteq \mathbb{R}^4$ to $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, where A and its reduced row echelon form are shown below:

$$A := \begin{bmatrix} 2 & -1 & 3 & 0 \\ 3 & 4 & -1 & 22 \\ -1 & 3 & -4 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hint: backsolve from the reduced row echelon form, write your explicit solution in linear combination form, and identify a basis.

Tuesday October 18

If we wish to find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$, then the following algorithm will always work: reduce the augmented matrix, backsolve and write the explicit solution in linear combination form. The vectors that you are taking linear combinations of will always span the solution space, by construction. If you follow this algorithm they will automatically be linearly independent, so they will be a basis for the solution space. This is illustrated in the large example below:

<u>Exercise 1</u> Consider the matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^6 \ s.t.\ A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

Solution (don't peek :-): backsolve, realizing that the augmented matrices have final columns of zero. Label the free variables with the letter "t", and subscripts to show the non-leading 1 columns from which they arose:

$$x_6 = t_6, x_5 = t_5, x_4 = t_4, x_3 = -2\,t_4 + t_5 - 3\,t_6$$
$$x_2 = t_2, x_1 = -2\,t_2 - t_4 - t_5 - 2\,t_6\,.$$

In vector form and then linear combination form this is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the four vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for the solution space:

- *They span the solution space by construction.*
- *They are linearly independent because if we set a linear combination equal to zero:*

$$t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}$$

then looking in the second entry implies $t_2 = 0$, the fourth entry implies $t_4 = 0$, and similarly $t_5 = t_6 = 0$.

Exercise 2) Focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. (If it helps, think of renaming the vector $\underline{\boldsymbol{x}}$ in the example above, with a vector $\underline{\boldsymbol{c}}$ of linear combination coefficients; then recall the prime Chapter 4 algebra fact that

$$A\,\underline{\boldsymbol{c}} = c_1 col_1(A) + c_2 col_2(A) + \dots + c_n col_n(A)$$

so any solution $\underline{\boldsymbol{c}}$ to $A\,\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$ is secretly a columns dependency, and vise-verse.)

Now, since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore your basis in Exercise 1 for the homogeneous solution space in $\mathbb{R}^6$ can also be thought of as a "basis" of the key column dependencies in $\mathbb{R}^4$, for both the original matrix, and for the reduced row echelon form.

2a) Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2b) Find a basis for $W =$ the span of the five columns of A. Hint: this will be a two-dimensional subspace of $\mathbb{R}^4$ and cyou can create it by successively removing redundant (dependent) vectors from the original collection of the five column vectors, until your remaining set still spans W but is linearly independent.

Exercise 3) (This exercise explains why any given matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier, back in Chapter 3.) Let $B_{4 \times 5}$ be a matrix whose columns satisfy the following dependencies:

$$col_1(B) \neq \underline{\mathbf{0}} \quad \text{(i.e. is independent)}$$
$$col_2(B) = 3\ col_1(B)$$
$$col_3(B) \text{ is independent of column 1}$$
$$col_4(B) \text{ is independent of columns 1,3.}$$
$$col_5(B) = -3\ col_1(B) + 2\ col_3(B) - col_4(B).$$

What is the reduced row echelon form of B?

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, as we return to studying differential equations tomorrow.

key fact: If n vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ span a subspace W then any collection $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular $W = \mathbb{R}^n$.)

Thus:

1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if vectors $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{\boldsymbol{w}}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent.)

4) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are in W and are linearly independent, they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\boldsymbol{w}_{n+1}$ not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.

* Check: If $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are linearly independent, and $\underline{\boldsymbol{w}}_{n+1} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$, then $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n, \underline{\boldsymbol{w}}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n + d\,\underline{\boldsymbol{w}}_{n+1} = \underline{\boldsymbol{0}}.$$

Since $\underline{\boldsymbol{w}} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{\boldsymbol{w}}_{n+1}$ as a linear combination of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$). But once $d = 0$, we have

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n = \underline{\boldsymbol{0}}$$

which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$.

Math 2250-010
Wed Oct 19

5.1 Second order linear differential equations, and vector space theory connections.

Definition: A vector space is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true.
(d) $\forall\ f \in V\ \exists\ -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $\left(c_1 + c_2\right) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot \left(c_2 \cdot f\right) = \left(c_1 c_2\right) \cdot f$ (associative property)
(h) $1 \cdot f = f,\ (-1) \cdot f = -f,\ 0 \cdot f = 0$ (these last two actually follow from the others).

Examples we've seen:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

Exercise 0) In Chapter 5 we focus on the vector space

$$V = C(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{R}\ s.t.\ f\ is\ a\ continuous\ function\}$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \ldots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

One reason this DE is called linear is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 1a) Check the linearity properties (1),(2) for the differential operator L .

1b) Use these properties to show that

Theorem 0: the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the "same" proof we used Monday to show that the solution space to a homogeneous matrix equation is a subspace.

Exercise 2) Find the solution space to homogeneous differential equation for $y(x)$

$$y'' + 2\,y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

Exercise 3) Use the linearity properties to show

Theorem 1: All solutions to the nonhomogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, but in the context of matrix equations, a week or two ago.)

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 4) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. The theorem and the techniques we'll actually be using going forward are illustrated by

Exercise 5) Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

5a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

5b) Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Then use the uniqueness theorem to deduce that y_1, y_2 are a basis for the solution space to this homogeneous differential equation, so that the solution space is indeed two-dimensional.

5c) Now consider the inhomogeneous DE

$$y'' - 2y' - 3y = 9$$

Notice that $y_P(x) = -3$ is a particular solution. Use this information and superposition (linearity) to solve the initial value problem

$$y'' - 2y' - 3y = 9$$
$$y(0) = 6$$
$$y'(0) = -2.$$

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2, we can use it to prove the dimension result Theorem 3. Here's how (and this is really just an abstractified version of the example on the previous page):

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by good guessing, as in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:
• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

we set

$$c_1 y_1(x_0) + c_2 y_2(x_0) = b_0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = b_1$$

which has unique solution $[c_1, c_2]^T$ given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus y_1, y_2 span the solution space.
• Linear independence: The computation above shows that there is only one way to write any solution $y(x)$ to the differential equation as a linear combination of y_1, y_2, because the linear combination coefficients c_1, c_2 are uniquely determined by the values of $y(x_0), y'(x_0)$. (In particular they must be zero if $y(x) \equiv 0$, because for the zero function b_0, b_1 are both zero so c_1, c_2 are too. This shows linear independence.)

Math 2250-004
Fri Oct 21
5.2: general theory for n^{th}-order linear differential equations; tests for linear independence;
also begin 5.3: finding the solution space to homogeneous linear constant coefficient differential equations by trying exponential functions as potential basis functions.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis $y_1, y_2, \ldots y_n$ for the n-dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$.
(2) If the DE is non-homogeneous, find a particular solution y_P. Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)
(3) Find values for the n free parameters $c_1, c_2, \ldots c_n$ in

$$y = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

to solve the initial value problem with initial values $b_0, b_1, \ldots, b_{n-1}$. (This last step just reduces to a matrix problem like in Chapter 3, where the matrix is the Wronskian matrix of $y_1, y_2, \ldots y_n$, evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0.)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically today and Monday. On Tuesday we'll begin the applications in section 5.4. We should have some fun experiments later next week to compare our mathematical modeling with physical reality.

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x)$ $\neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I .)

This DE is called linear because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y)$, $c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n = 2, that we checked Wednesday. Then, since the $y = y_P + y_H$ proof only depended on the linearity properties of L, just like yesterday, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

is a subspace.

Theorem 1: The general solution to the nonhomogeneous n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x),\ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$
\begin{gathered}
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\
y(x_0) = b_0 \\
y'(x_0) = b_1 \\
y''(x_0) = b_2 \\
\vdots \\
y^{(n-1)}(x_0) = b_{n-1}
\end{gathered}
$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case n = 2, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1y_1 + c_2y_2 + \ldots + c_ny_n.$$

proof: By the existence half of Theorem 2, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots. y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$
\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix},
\begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots,
\begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}
$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ span the solution space. If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then because the zero function has zero initial vector $[b_0, b_1, \ldots b_{n-1}]^T$ the matrix equation above implies that $[c_1, c_2, \ldots c_n]^T = \underline{\mathbf{0}}$, so $y_1, y_2, \ldots y_n$ are also linearly independent. Thus, $y_1, y_2, \ldots y_n$ are a basis for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

□

Let's do some new exercises that tie these ideas together.

Exercise 1) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$y''' + 3\,y'' - y' - 3\,y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Make sure to use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

Exercise 2a) Find the general solution to

$$y''' + 3\,y'' - y' - 3\,y = 6\,.$$

Hint: First try to find a particular solution ... try a constant function.

b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1,\, y'(0) = 2,\, y''(0) = 7$.

for fun now, but maybe not just for fun later:

```
> with(DEtools) :
  dsolve({y'''(x) + 3·y''(x) − y'(x) − 3·y(x) = 6, y(0) = −1, y'(0) = 2, y''(0) = 7});
```

• In section 5.2 there is a focus on testing whether collections of functions are linearly independent or not. This is important for finding bases for the solution spaces to homogeneous linear DE's because of the fact that if we find n linearly independent solutions to the n^{th} order homogeneous DE, they will automatically span the n-dimensional the solution space. (We discussed this general vector space "magic" fact on Wednesday.) And checking just linear independence is sometimes easier than also checking the spanning property.

Ways to check whether functions $y_1, y_2, \ldots y_n$ are linearly independent on an interval:

In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

where "0" is the zero function which equals 0 for all x on our interval.

Method 1) Plug in different $x-$ values to get a system of algebraic equations for $c_1, c_2 .. c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = \ldots = c_n = 0$, or you'll find a likely dependency.

Exercise 3) Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \ y_2(x) = x, \ y_3(x) = x^2$$

are linearly independent. (These functions show up in the homework due Monday.) For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 4) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\, x \in I$, then we can take derivatives to get a system

$$\begin{array}{l} c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0 \\ c_1 y_1{}' + c_2 y_2{}' + \ldots + c_n y_n{}' = 0 \\ c_1 y_1{}'' + c_2 y_2{}'' + \ldots + c_n y_n{}'' = 0 \end{array}$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} = 0$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 5) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2$$

are linearly independent. Use $x_0 = 1$.

Remark 1) Method 3 is usually not the easiest way to prove independence. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve matrix algebra problem

$$\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

for the vector $[c_1, c_2, \ldots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the homogeneous solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations uniquely solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\, x \in I$, or zero $\forall\, x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 6) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\, x \in \mathbb{R}$.