Math 2250-004
Week 5 notes: Sections 3.2-3.5
3.2 Gaussian elimination for solving systems of linear algebraic equations
3.3 Structure of solutions sets for systems of linear equations
3.4 Matrix algebra
3.5 Matrix inverses

Monday September 19
3.1-3.3 Linear systems of (algebraic) equations and how to solve them via Gaussian elimination and the reduced row echelon form of augmented matrices.

- Discuss the remaining parts of last Friday's notes. As we solve those systems of linear algebraic equations we will begin to see how to systematically approach the problem of finding the explicit solution set. The precise details are below, and they should make good sense after we do the Friday examples. There is also a larger example in today's notes.

....
Summary of the systematic method known as Gaussian elimination:

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, \ldots x_n\right]^T$ of the n unknowns as

$$
\begin{array}{ccccccccc}
a_{11}\,x_1 & + & a_{12}\,x_2 & + \ldots + & a_{1\,n}\,x_n & = & b_1 \\
a_{21}\,x_1 & + & a_{22}\,x_2 & + \ldots + & a_{2\,n}\,x_n & = & b_2 \\
\vdots & & & & \vdots & & \vdots \\
a_{m1}\,x_1 & + & a_{m2}\,x_2 & + \ldots + & a_{m\,n}\,x_n & = & b_m
\end{array}
$$

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = \left[a_{i\,j}\right]$ is called the *augmented matrix* $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$
\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} & b_1 \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} & b_2 \\
\vdots & & & & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} & b_m
\end{array}\right]
$$

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the *solution set* .

There are three types of *elementary equation operations* that don't change the solution set to the linear system. They are

- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of *elementary row operations:*

- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in

row-echelon form:
(1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.
(2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it.

(At this stage you could "backsolve" to find all solutions.)

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with
(3) Each leading non-zero row entry has value 1. Such entries are called "leading $1's$ "
(4) Each column that has (a row's) leading 1 has $0's$ in all the other entries.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. All mathematical software will have a command to find the reduced row echelon form of a matrix.

Exercise 1 Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27 . \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1, −2, 3, 2, 1,
                     2, −4, 8, 3, 10,
                     3, −6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  ⟨A|b⟩;      # the mathematical augmented matrix doesn't actually have
               # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(⟨A|b⟩);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \tag{1}$$

```
> LinearSolve(A, b);
        #  this command will actually write down the general solution, using
        # Maple's way of writing free parameters, which actually makes
        # some sense.  Generally when there  are free parameters involved,
        #  there will be equivalent ways to express the solution that may
        # look different.  But usually Maple's version will look like yours,
        # because it's using the same algorithm and choosing the free parameters
        # the same way too.

>
```

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* finish the last exercise from yesterday's notes if necessary, then:

3.3 The structure of the solution sets to linear algebraic systems of equations, based on reduced row echelon form computations.

Exercise 1a What four conditions are necessary for a matrix to be in reduced row echelon form?

Exercise 1b Are the following matrices in reduced row echelon form or not? Explain.

a)

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Exercise 2 Coefficient matrix taken from problem #19, section 3.3, page 174.

```
> with(LinearAlgebra) :
> A := Matrix(3, 5, [2, 7, -10, -19, 13,
                     1, 3, -4, -8, 6,
                     1, 0, 2, 1, 3]);
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \tag{2}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$\left[\begin{array}{ccccc|ccc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right].$$

We'll try solving three linear systems at once!

```
> b1 := Vector([0, 0, 0]) :
  b2 := Vector([7, 0, 0]) :
  b3 := Vector([7, 3, 0]) :
  C := ⟨A|b1|b2|b3⟩; # very augmented matrix
  ReducedRowEchelonForm(C);
```

$$C := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4}$$

2a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

Important conceptual questions:

2b) Which of these three solutions could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix and the reduced row echelon form of the augmented matrix? Why?

2c) Linear systems in which right hand side vectors equal zero are called homogeneous linear systems. Otherwise they are called inhomogeneous or nonhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? It's related to an important general concept which will keep coming up in the rest of the course.

Exercise 3) The reduced row echelon form of a (non-augmented) matrix A can tell us a lot about the possible solution sets to linear systems with augmented matrices $\langle A|\underline{\boldsymbol{b}}\rangle$.

First (notation), recall that linear systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be written more efficiently using the rule we use to multiply a matrix times a vector,

$$\boldsymbol{A}\,\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

so that they system above can be abbreviated by $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$.

Then consider the matrix A below, and answer all questions:

```
> A := Matrix(2, 5, [2, 7,-10,-19, 13, 1, 3,-4,-8, 6]);
  ReducedRowEchelonForm(A);
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \end{bmatrix} \qquad (5)$$

3a) Is the homogeneous problem $A\underline{\boldsymbol{x}}=\underline{\boldsymbol{0}}$ always solvable?

3b) Is the inhomogeneous problem $A\underline{\boldsymbol{x}}=\underline{\boldsymbol{b}}$ solvable no matter the choice of $\underline{\boldsymbol{b}}$?

3c) How many solutions are there? How many free parameters are there in the solution? How does this number relate to the reduced row echelon form of A?

Exercise 4) Now consider the matrix B and similar questions:

> $B := Matrix(3, 2, [1, 2, -1, 3, 4, 2]);$
> $ReducedRowEchelonForm(B);$

$$B := \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \textbf{(6)}$$

4a) How many solutions to the homogeneous problem $B\underline{\mathbf{x}} = \underline{\mathbf{0}}$?

4b) Is the inhomogeneous problem $B\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ solvable for every right side vector $\underline{\boldsymbol{b}}$?

4c) When the inhomogeneous problem is solvable, how many solutions does it have?

Exercise 5) Square matrices (i.e number of rows equals number of columns) with 1's down the diagonal which runs from the upper left to lower right corner are special. They are called identity matrices, I (because $I\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$ is always true (as long as the vector $\underline{\boldsymbol{x}}$ is the right size)).

```
> C := Matrix(4, 4, [1, 0, -1, 1, 22, -1, 3, 5, 7, 4, 6, 2, 3, 5, 7, 13]);
  ReducedRowEchelonForm(C);
```

$$C := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 22 & -1 & 3 & 5 \\ 7 & 4 & 6 & 2 \\ 3 & 5 & 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

```
>
```

5a) How many solutions to the homogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$?

5b) Is the inhomogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ solvable for every choice of $\underline{\boldsymbol{b}}$?

5c) How many solutions?

Exercise 6: What are your general conclusions?

6a) What conditions on the reduced row echelon form of the matrix A guarantee that the homogeneous equation $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has infinitely many solutions?

6b) What conditions on the dimensions of A (i.e. number of rows and number of columns) always force infinitely many solutions to the homogeneous problem?

6c) What conditions on the reduced row echelon form of A guarantee that solutions $\underline{\mathbf{x}}$ to $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ are always unique (if they exist)?

6d) If A is a square matrix ($m=n$), what can you say about the solution set to $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ when

- The reduced row echelon form of A is the identity matrix?
- The reduced row echelon form of A is not the identity matrix?

Wednesday Sept 21
3.4 Matrix algebra

- Our first exam is next Friday September 30 You have a homework assignment covering 3.5-3.6 which is due a week from today, on Wednesday September 28, and which will be posted on our homework page by later today. (The 3.1-3.4 homework is due this Friday.) The exam will cover through 3.6.

- Vector and matrix algebra, section 3.4:

Matrix vector algebra that we've already touched on, but that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ : \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ : \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := x_1\, y_1 + x_2\, y_2 + \ldots + x_n\, y_n .$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ : & & & & : \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\,n}x_n \\ : \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\mathbf{x}} \\ Row_2(A) \cdot \underline{\mathbf{x}} \\ : \\ Row_m(A) \cdot \underline{\mathbf{x}} \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}} .$$

Exercise 1a) Compute

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

Exercise 1b): Check that the vector dot product distributes over vector addition and scalar multiplication, i. e.

$$\underline{\boldsymbol{a}} \bullet (\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) = \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{x}} + \underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{y}}$$
$$\underline{\boldsymbol{a}} \bullet (c\,\underline{\boldsymbol{x}}) = c(\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{x}}) \,.$$

Since the dot product is commutative or by checking directly we also deduce

$$(\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) \bullet \underline{\boldsymbol{a}} = \underline{\boldsymbol{x}} \bullet \underline{\boldsymbol{a}} + \underline{\boldsymbol{y}} \bullet \underline{\boldsymbol{a}}$$
$$(c\,\underline{\boldsymbol{a}}) \bullet \underline{\boldsymbol{x}} = c(\underline{\boldsymbol{a}} \bullet \underline{\boldsymbol{x}}) \,.$$

Exercise 2) Use your work from b to show that matrix multiplication distributes over vector addition and scalar multiplication, i.e.

$$A(\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) = A\,\underline{\boldsymbol{x}} + A\,\underline{\boldsymbol{y}}$$
$$A(c\,\underline{\boldsymbol{x}}) = c\,A\,\underline{\boldsymbol{x}}$$

Do this by comparing the i^{th} entries of the vectors on the left, to those on the right. For any vector

$$\underline{\boldsymbol{b}} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}$$

we will use the notation $entry_i(\underline{\boldsymbol{b}})$ for b_i .

Matrix algebra:

- addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{i\,j}(A) = a_{i\,j}$, $entry_{i\,j}(B) = b_{i\,j}$. (In this case we write $A = [a_{i\,j}]$, $B = [b_{i\,j}]$.) Let c be a scalar. Then

$$entry_{i\,j}(A + B) := a_{i\,j} + b_{i\,j}.$$
$$entry_{i\,j}(c\,A) := c\,a_{i\,j}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

- <u>matrix multiplication:</u> Let $A_{m \times n}$, $B_{n \times p}$ be two matrices such that the number of columns of A equals the number of rows of B. Then the product AB is an $m \times p$ matrix, with

$$entry_{ij}(AB) := row_i(A) \cdot col_j(B) = \sum_{k=1}^{n} a_{ik} b_{kj} \,.$$

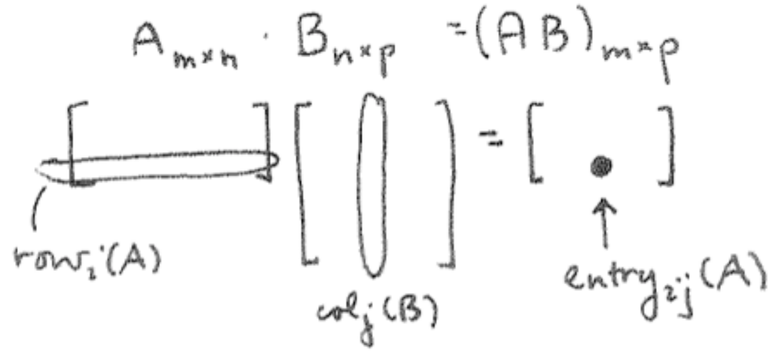
Equivalently, the j^{th} column of AB is given by the matrix times vector product

$$col_j(AB) = A \; col_j(B)$$

and the i^{th} row of AB is given by the product

$$row_i(AB) = row_i(A) \; B.$$

This stencil might help:



<u>Exercise 4)</u>

<u>a)</u> Can you compute AB for the matrices A, B in exercise 3?

b) Let $C := \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Using $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ compute AC and CA and check the row and column properties above. Also notice that $AC \neq CA$, and the sizes of these two product matrices aren't even the same.

Properties for the algebra of matrix addition and multiplication :

- Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $\quad A + B = B + A$
- $+$ is associative $\quad (A + B) + C = A + (B + C)$
- scalar multiplication distributes over $+$ $\quad c(A + B) = cA + cB$.
- multiplication is associative $\quad (AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $\quad A(B + C) = AB + AC$;
  $(A + B)C = AC + BC$

Exercise 5:

a) Verify some of these properties in general - except for the associative property for multiplication they're all easy to check.

b) For the multiplicative associative property verify that at least the dimensions of the triple product matrices are the same.

c) Then check that for the matrices in exercises 3-4, it is indeed true that $(AC)B = A(CB)$.

Friday September 23

3.5 Matrix inverses.

We've been talking about matrix algebra: addition, scalar multiplication, multiplication, and how these operations combine. If necessary, finish those notes. Here's part of the last exercise, filled in:

<u>Properties for the algebra of matrix addition and multiplication :</u>

- Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $\quad A + B = B + A$

$$entry_{ij}(A + B) = a_{ij} + b_{ij} = b_{ij} + a_{ij} = entry_{ij}(B + A)$$

- $+$ is associative $\quad (A + B) + C = A + (B + C)$

the ij entry of each side is $a_{ij} + b_{ij} + c_{ij}$

- scalar multiplication distributes over $+$ $\quad c(A + B) = cA + cB$.

ij entry of LHS is $c\left(a_{ij} + b_{ij}\right) = c\left(a_{ij} + b_{ij}\right) = ij$ *entry of RHS*

- multiplication is associative $\quad (AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $\quad A(B + C) = AB + AC$;

ij entry of LHS $= row_i(A) \cdot col_j(B + C)$

$row_i(A) \cdot col_j(B) + row_i(A) \cdot col_j(C) =$ ij entry of RHS.

$$(A + B)C = AC + BC$$

But I haven't told you what the algebra on the previous page is good for. Today we'll start to find out. By way of comparison, think of a scalar linear equation with known numbers a, b, c, d and an unknown number x,

$$a\,x + b = c\,x + d$$

We know how to solve it by collecting terms and doing scalar algebra:

$$a\,x - c\,x = d - b$$

$$(a - c)\,x = d - b \qquad *$$

$$x = \frac{d - b}{a - c}\,.$$

How would you solve such an equation if A, B, C, D were square matrices, and X was a vector (or matrix) ? Well, you could use the matrix algebra properties we've been discussing to get to the $*$ step. And then if X was a vector you could solve the system * with Gaussian elimination. In fact, if X was a matrix, you could solve for each column of X (and do it all at once) with Gaussian elimination.

But you couldn't proceed as with scalars and do the final step after the * because it is not possible to divide by a matrix. Today we'll talk about a potential shortcut for that last step that is an analog of of dividing, in order to solve for X. It involves the concept of *inverse matrices.*

Step 1:
<u>Identity matrices:</u> Recall that the $n \times n$ identity matrix $I_{n \times n}$ has one's down the diagonal (by which we mean the diagonal from the upper left to lower right corner), and zeroes elsewhere. For example,

$$I_{1 \times 1} = [1]\,, \quad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

In other words, $entry_{i\,i}\left(I_{n \times n}\right) = 1$ and $entry_{i\,j}\left(I_{n \times n}\right) = 0$ if $i \neq j$.

<u>Exercise 1)</u> Check that

$$A_{m \times n} I_{n \times n} = A, \qquad I_{m \times m} A_{m \times n} = A\,.$$

Hint: check that the matrices on each side are the same size, and that each of their $i - j$ entries agree.

<u>Remark</u>: That's why these matrices are called <u>identity matrices</u> - they are the matrix version of multiplicative identities, e.g. like multiplying by the number 1 in the real number system.)

Step 2:

<u>Matrix inverses:</u> A square matrix $A_{n \times n}$ is <u>invertible</u> if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I .$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

<u>Remark:</u> A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

<u>Exercise 2a)</u> Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Inverse matrices are very useful in solving algebra problems. For example

<u>Theorem:</u> If A^{-1} exists then the only solution to $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

<u>Exercise 2b)</u> Use the theorem and A^{-1} in <u>2a</u>, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

Exercise 3a) Use matrix algebra to verify why the Theorem on the previous page is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\,A^{-1}$ (this second product can't even be computed because the dimensions don't match up!).

3b) Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, solve for X in terms of the other matrices:

$$XA + C = B$$

Step 3:

But where did that formula for A^{-1} come from?

Answer: Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I.$$

We can break this matrix equation down by the columns of *X*. In the two by two case we get:

$$A\left[\begin{array}{c|c} col_1(X) & col_2(X) \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In other words, the two columns of the inverse matrix *X* should satisfy

$$A\,(col_1(X)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\,(col_2(X)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once, as we've done before when we had different right hand sides:

Exercise 4: Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

Exercise 5: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}?$$

Exercise 6) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Hint: We'll discover that it's impossible for B to have an inverse.

Exercise 7) What happens when we ask software like Maple for the inverse matrices above?

> $with(LinearAlgebra):$

> $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}^{-1};$

> $\begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}^{-1};$

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated on the previous page will always yield the inverse matrix.

explanation: By the previous theorem, when A^{-1} exists, the solutions to linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

are unique ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). From our discussions on Tuesday and Wednesday, we know that for square matrices, solutions to such linear systems exist and are unique only if the reduced row echelon form of A is the identity. (Do you remember why?) Thus by logic, whenever A^{-1} exists, A reduces to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e.

$$A\left[\begin{array}{c|c|c|c} col_1(X) & col_2(X) & \cdots & col_n(X) \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & 1 \end{array}\right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we just worked, by using a chain of elementary row operations:

$$[\,A\,|\,I\,] \to \to \to \to \to [\,I\,|\,B\,],$$

and deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I\,.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[\,I\,|\,B\,]$ to read $[\,B\,|\,I\,]$ and then reversing each of the elementary row operations in the first computation, i.e. create the chain

$$[\,B\,|\,I\,] \to \to \to \to \to [\,I\,|\,A\,]\,.$$

so $B\,A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then solutions $\underline{\boldsymbol{x}}$ to $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ always exist and are unique, so the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists. That's exactly what the Theorem claims.

There's a nice formula for the inverses of 2×2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant $D = ad - bd$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 8a) Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

8b) Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$\begin{aligned} 3x + 7y &= 5 \\ 5x + 4y &= 8\,. \end{aligned}$$

Remark: For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the reduced row echelon form will be the identity if and only if the two rows are not multiples of each other. If a, b are both non-zero this is equivalent to saying that the ratio of the first entries in the rows $\frac{c}{a} \neq \frac{d}{b}$, the ratio of the second entries. Cross multiplying we see this is the same as $ad \neq bc$, i.e. $ad - bc \neq 0$. This is also the correct condition for the rows not being multiples, even if one or both of a, b are zero, and so by the previous theorem this is the correct condition for knowing the inverse matrix exists.

Remark: Determinants are defined for square matrices $A_{n \times n}$ and they determine whether or not the inverse matrices exist, (i.e. whether the reduced row echelon form of A is the identity matrix). And when $n > 2$ there are analogous (more complicated) magic formulas for the inverse matrices, that generalize the one above for $n = 2$. This is section 3.6 material that we'll discuss carefully on Monday and Tuesday.