

Math 2250-004 : Week 2, material from sections 1.3, 1.4, 1.5, EP 3.7

Monday August 29

We will mostly use last Friday's notes. Our goals today are

- (1) understand what makes a first order differential equation separable.
- (2) understand the algorithm based on differentials that solves separable differential equations: why it works, and how it sometimes misses "singular solutions"
- (3) understand and apply the existence-uniqueness theorem for first order DE initial value problems.

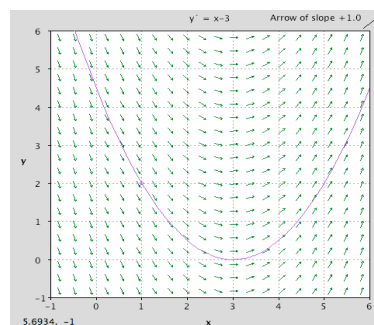
When discussing the existence-uniqueness theorem at the end of Friday's notes today, we'll refer to examples from Wednesday's notes that we discussed on Friday. Those were:

Exercise 1 (Wednesday):

$$\frac{dy}{dx} = x - 3$$

$$y(1) = 2.$$

We found the solution $y(x) = \frac{x^2}{2} - 3x + \frac{9}{2} = \frac{(x-3)^2}{2}$. Is this consistent with the existence-uniqueness theorem?



Exercise 2: (Wednesday)

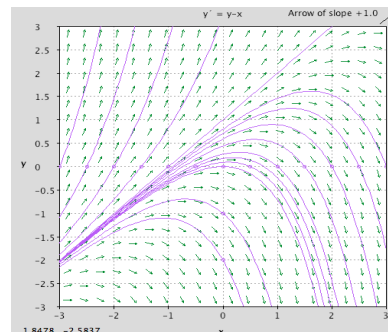
$$\frac{dy}{dx} = y - x$$

$$y(0) = 0$$

From a family of solutions that was given to us, we found a solution

$$y(x) = x + 1 - e^x.$$

Is this the only possible solution? Hint: use the existence-uniqueness theorem.



Exercise 1 (today): Here's another example of using a separable DE to illustrate the existence-uniqueness theorem.

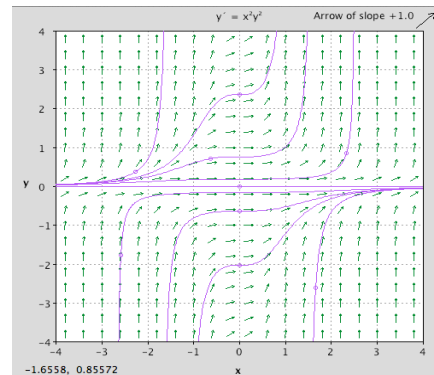
a) Does each IVP

$$y' = x^2 y^2$$

$$y(x_0) = y_0$$

have a unique solution?

b) Find all solutions to this differential equation.



Maple check (notice it misses the singular solution):

```
> with(DEtools) :
  dsolve(y'(x) = x^2 * y(x)^2, y(x));
```

$$y(x) = \frac{3}{-x^3 + 3_CI}$$

(1)

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = x^2 + y^2$$

$$y(x_0) = y_0$$

b)

$$y' = x^4 + y^4$$

$$y(x_0) = y_0$$

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Tues Aug 30

1.4: separable DEs, examples and experiment.

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week, e.g. exponential growth/decay and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

- In order to solidify the ideas of the existence-uniqueness theorem from Friday's notes, along with the process for solving separable differential equations, work through Monday's notes. We will also refer to Friday's notes so have them handy as well.

Toricelli's Law, for draining water tanks. Refer to the figure below.

Exercise 1:

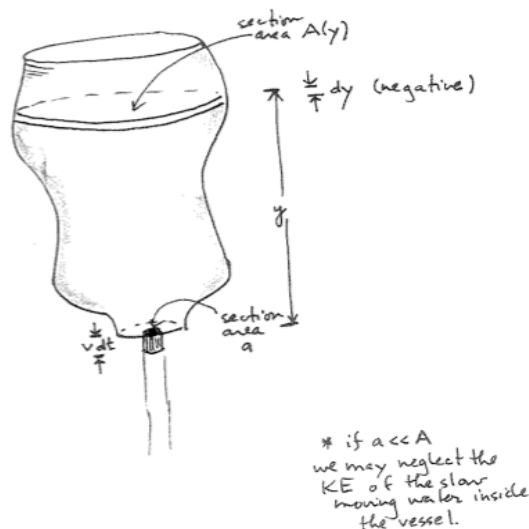
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligible velocity ($a \ll A$). Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$v = \sqrt{2gy}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y) \frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2g}).$$



Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A , so the Toricelli DE and IVP becomes

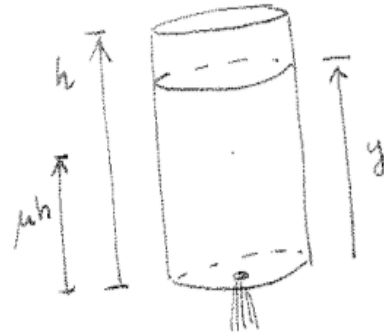
$$\frac{dy}{dt} = -k y^{\frac{1}{2}}$$

$$y(0) = h$$

(different k). Let $T(\mu h)$ be the time it takes the the water to go from height h (full) to height $\mu \cdot h$, where the fraction μ is between 0 and 1. Note, $T(h) = 0$ and $T(0)$ is the time it takes for the tank to empty completely.

Exercise 2: (We will use this calculation in our experiment) Solve the IVP and then show the time it takes the tank to empty completely is related to $T(\mu h)$ by

$$T(0) = \frac{T(\mu h)}{1 - \sqrt{\mu}}.$$



Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
[> Digits := 5 : # that should be enough significant digits
[> 1 / (1 - sqrt(.5)) ; # the factor from above, when mu is 0.5
[> Thalf := 35; # seconds to half-empty canteen
[> Tpredict := 3.4143 * Thalf; #prediction
[>
```

Remark: Maple can draw direction fields, although they're not as easy to create as in "dfield". On the other hand, Maple can do any undergraduate mathematics computation, including solving pretty much any differential equation that has a closed form solution. Let's see what the commands below produce. We'll get some error messages related to the existence-uniqueness theorem!

```
> with(DEtools) : # this loads a library of DE commands
```

```
> deqtn1 := y'(t) = -sqrt(y(t)); # took k=1
ics1 := y(0) = 1; # took initial height=1
```

$$\begin{aligned} \text{deqtn1} &:= D(y)(t) = -\sqrt{y(t)} \\ \text{ics1} &:= y(0) = 1 \end{aligned} \quad (2)$$

```
> dsolve({deqtn1, ics1}, y(t)); # DE IVP sol!
```

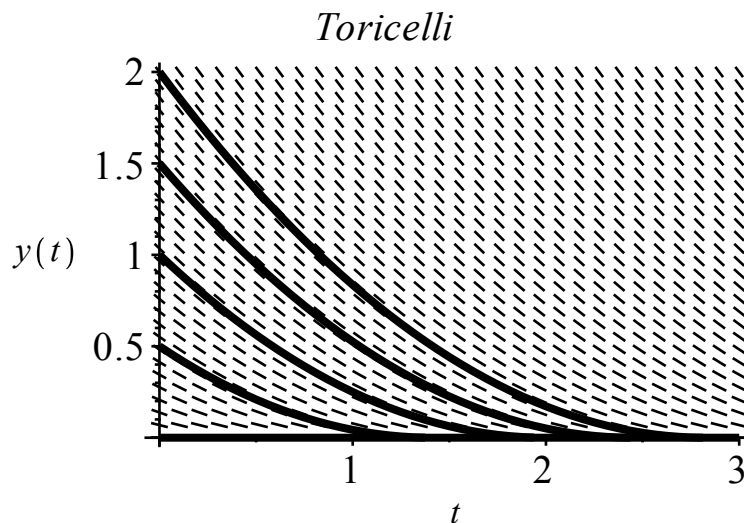
$$y(t) = \frac{1}{4} t^2 - t + 1 \quad (3)$$

```
> factor(%); #how we wrote it
```

$$y(t) = \frac{1}{4} (t - 2)^2 \quad (4)$$

```
> DEplot(deqtn1, y(t), 0..3, {[y(0) = 0], [y(0) = 1.], [y(0) = 2.], [y(0) = 0.5], [y(0) = 1.5]},
arrows = line, color = black, linecolor = black, dirgrid = [30, 30], stepsize = .1, title
= `Toricelli`);
```

Warning, plot may be incomplete, the following errors(s) were issued:
cannot evaluate the solution further right of 1.4141924, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
cannot evaluate the solution further right of 1.9999776, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
cannot evaluate the solution further right of 2.4494671, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
cannot evaluate the solution further right of 2.8283979, probably a singularity



Math 2250-004

Wed Aug31

1.5: linear DEs, and applications.

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x)$$

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

a) $y'(x) = -2y + 4x^2$

b) $y'(x) = x - y^2 + 1$

c) $y'(x) = x^2 - x^2y + 1$

d) $y'(x) = \frac{6x - 3xy}{x^2 + 1}$

e) $y'(x) = x^2 + y^2$

f) $y'(x) = x^2e^{x^3}$.

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P . Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$e^{\int P(x)dx} (y' + P(x)y) = e^{\int P(x)dx} Q(x)$$

because the left side is a derivative (product rule):

$$\frac{d}{dx} \left(e^{\int P(x)dx} y \right) = e^{\int P(x)dx} Q(x).$$

So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} Q(x)dx + C.$$

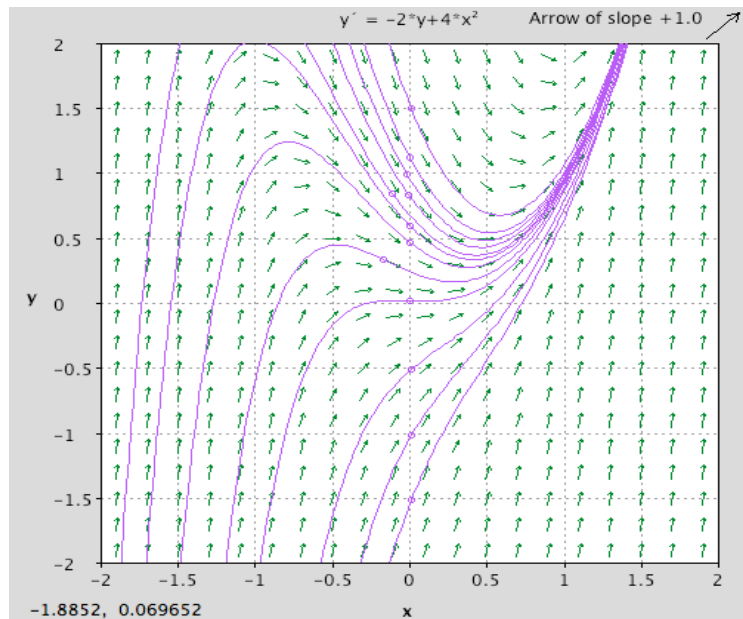
Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

Exercise 2: Solve the differential equation

$$y'(x) = -2y + 4x^2,$$

and compare your solutions to the dfield plot below.

```
> with(DEtools): # load differential equations library
> deqtn2 := y'(x) = -2*y(x) + 4*x^2: #notice you must use · for multiplication in Maple,
# and write y(x) rather than y.
dsolve(deqtn2, y(x)); # Maple check
>
```

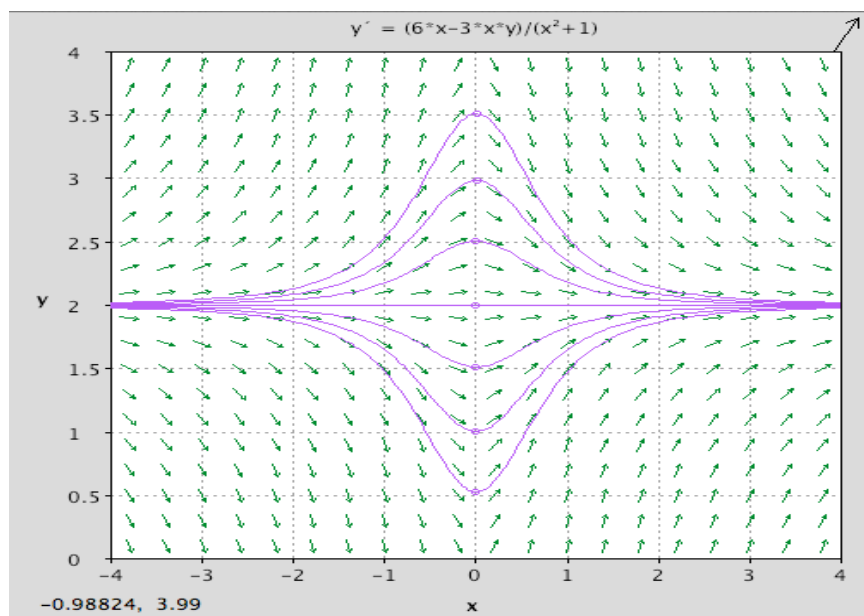


Exercise 3: Find all solutions to the linear (and separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C(x^2 + 1)^{-\frac{3}{2}}$.

`> with(DEtools) :`
`dsolve(`

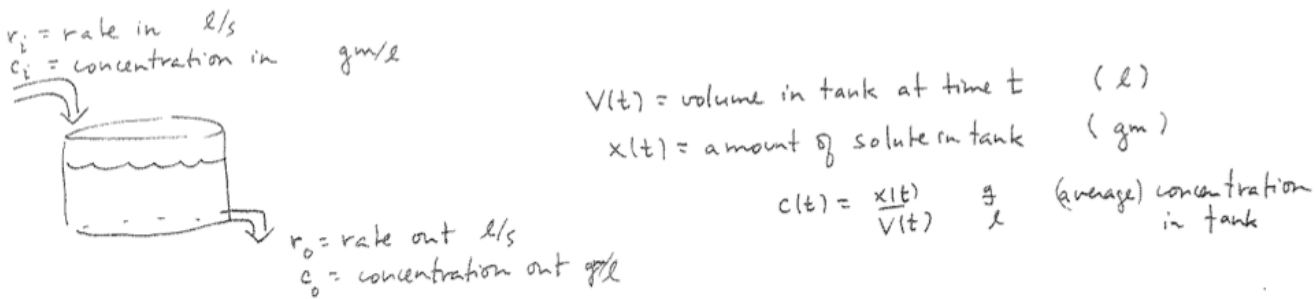


An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_o . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \frac{gm}{l}.$$

See the diagram below.



Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_o$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_o(\tau) d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a x = b$$

$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.

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Fri Sept 2

1.5: linear DEs, and applications.

Recall from Wednesday notes that input-out models often lead to the IVP

$$\begin{aligned}x' + a x &= b \\ x(0) &= x_0\end{aligned}$$

where a, b are (positive) constants.

Exercise 1: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a} \right) e^{-a t}.$$

Exercise 2: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron has a pretty constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the inflow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?



<http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF>

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

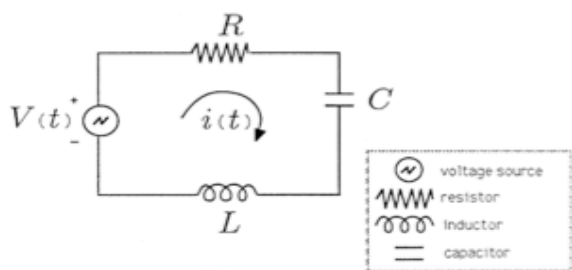
$$V = 480 \text{ km}^3$$

$$r = 350 \frac{\text{km}^3}{\text{y}}$$

b) Solve the IVP, and then answer the question.

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple RLC circuit modeling.



circuit element	voltage drop	units
inductor	$L I'(t)$	L Henries (H)
resistor	$R I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C} Q(t)$	C Farads (F)

<http://cnx.org/content/m21475/latest/pic012.png>

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.

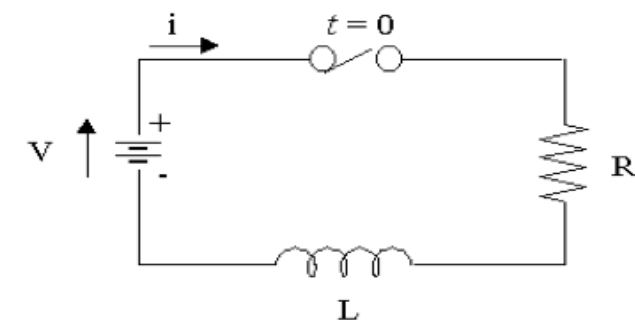
Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

$$\text{For } Q(t): \quad L Q''(t) + R Q'(t) + \frac{1}{C} Q(t) = V(t)$$

$$\text{For } I(t): \quad L I''(t) + R I'(t) + \frac{1}{C} I(t) = V'(t)$$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1^{st} order linear DE's, as below:

Exercise 3: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



<http://www.intmath.com/differential-equations/5-rl-circuits.php>