

Math 2250-004

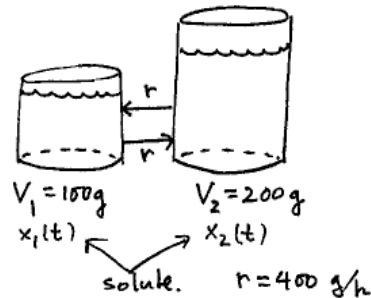
Week 14: sections 7.1-7.3 first order systems of linear differential equations

Mon Nov 28

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:

http://en.wikipedia.org/wiki/Multi-compartment_model

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t)$, $x_2(t)$ above, using input-output modeling.

Assume solute concentration is uniform in each tank. If $x_1(0) = b_1$, $x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

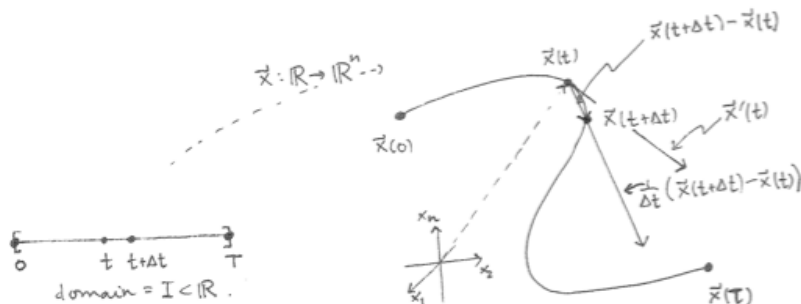
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\begin{aligned}\mathbf{x}'(t) &= \mathbf{F}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\mathbf{x}(t)$:

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\mathbf{x}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\mathbf{x}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\mathbf{x}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\begin{aligned}\mathbf{x}'(t) &:= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t + \Delta t) \\ x_2(t + \Delta t) \\ \vdots \\ x_n(t + \Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right) \\ &= \lim_{\Delta t \rightarrow 0} \begin{bmatrix} \frac{1}{\Delta t} (x_1(t + \Delta t) - x_1(t)) \\ \frac{1}{\Delta t} (x_2(t + \Delta t) - x_2(t)) \\ \vdots \\ \frac{1}{\Delta t} (x_n(t + \Delta t) - x_n(t)) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},\end{aligned}$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\mathbf{x}(t_0) = \mathbf{x}_0$), and you know your

"velocity" vector (depending on time and current location) \Rightarrow you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! You just convert the scalar quantities in the code into vector quantities. And this is what numerical solvers do.)

Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4x_1 + 2x_2$$

$$x_2'(t) = 4x_1 - 2x_2$$

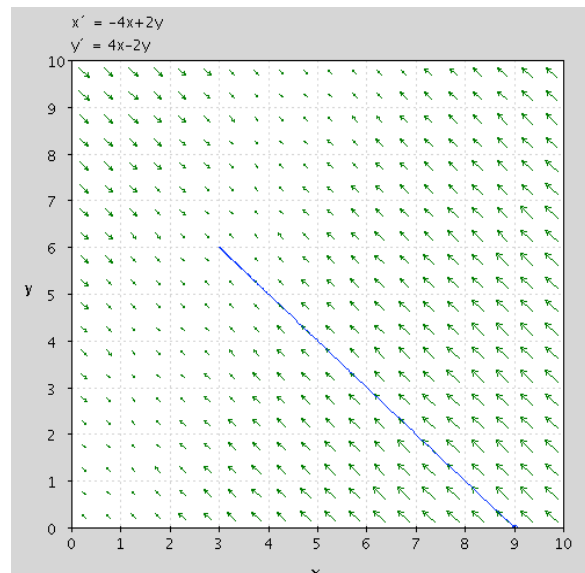
$$x_1(0) = 9$$

$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time t . If your system involved ten coupled tanks rather than two, then this "particle" is moving around in \mathbb{R}^{10} .

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



First order systems of differential equations of the form

$$\mathbf{x}'(t) = A \mathbf{x}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\mathbf{x}'(t) - A \mathbf{x} = \mathbf{0}$$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\mathbf{x}'(t) - A \mathbf{x} = \mathbf{f}(t)$$

or

$$\mathbf{x}'(t) = A \mathbf{x} + \mathbf{f}(t)$$

Notice that the operator on vector-valued functions $\mathbf{x}(t)$ defined by

$$L(\mathbf{x}(t)) := \mathbf{x}'(t) - A \mathbf{x}(t)$$

is linear, i.e.

$$L(\mathbf{x}(t) + \mathbf{y}(t)) = L(\mathbf{x}(t)) + L(\mathbf{y}(t))$$

$$L(c \mathbf{x}(t)) = c L(\mathbf{x}(t)).$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\mathbf{x}'(t) - A \mathbf{x} = \mathbf{0}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\mathbf{x}'(t) - A \mathbf{x} = \mathbf{f}(t)$$

will be of the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_H$$

where \mathbf{x}_p is any single particular solution and \mathbf{x}_H is the general homogeneous solution.

Exercise 3) In the case that A is a constant matrix (i.e. entries don't depend on t), consider the homogeneous problem

$$\mathbf{x}'(t) = A \mathbf{x}.$$

Look for solutions of the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v},$$

where \mathbf{v} is a constant vector. Show that $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ solves the homogeneous DE system if and only if \mathbf{v} is an eigenvector of A , with eigenvalue λ , i.e. $A \mathbf{v} = \lambda \mathbf{v}$.

Hint: In order for such an $\mathbf{x}(t)$ to solve the DE it must be true that

$$\mathbf{x}'(t) = \lambda e^{\lambda t} \mathbf{v}$$

and

$$A \mathbf{x}(t) = A e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v}$$

Set these two expressions equal.

Exercise 4) Use the idea of Exercise 3 to solve the initial value problem of Exercise 2!! Compare your solution $\underline{x}(t)$ to the parametric curve drawn by pplane, that we looked at a couple of pages back.

Exercise 5) Lessons learned from tank example: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\begin{aligned}\underline{x}'(t) &= A \underline{x} \\ \underline{x}(0) &= \underline{x}_0 \in \mathbb{R}^n\end{aligned}$$

using the method in Exercise 3-4 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)

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Tues Nov 29

Continue discussing systems of differential equations, 7.1-7.3

- Review exercises 3-5 in Monday's notes, which foreshadow the eigenvalue-eigenvector method for solving homogeneous first order systems of differential equation IVPs

$$\mathbf{x}'(t) = A \mathbf{x}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

that the book covers in detail in section 7.3.

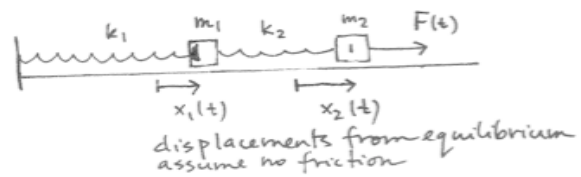
- Then discuss the important fact that every n^{th} order differential equation or system of differential equations is actually equivalent to a (possibly quite large) system of first order differential equations. This discussion is in today's notes and is the content of section 7.1. The existence-uniqueness theorem for first order systems of differential equations will then explain why the natural initial value problems for higher order differential equations, that we studied in Chapter 5, also always have unique solutions. Another mystery that this correspondence will solve is why we used "characteristic polynomial" when looking for basis solutions e^{rt} to n^{th} order constant coefficient homogeneous differential equations and then used exactly the same terminology, "characteristic polynomial", when we were looking for matrix eigenvalues (see section 7.3).

- Along the way and tomorrow as well, we will also be discussing how the theory and template for finding solutions to first order systems of linear differential equations precisely mirrors the template we developed for single n^{th} -order linear differential equations in Chapter 5. This is the content of section 7.2.

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

Example:

Consider this configuration of two coupled masses and springs:



Exercise 1) Use Newton's second law to derive a system of two second order differential equations for $x_1(t)$, $x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 2) Consider the IVP from Exercise 1, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40 \sin(3 t)$:

$$\begin{aligned}x_1'' &= -3 x_1 + x_2 \\x_2'' &= 2 x_1 - 2 x_2 + 40 \sin(3 t) \\x_1(0) &= b_1, x_1'(0) = b_2 \\x_2(0) &= c_1, x_2'(0) = c_2 .\end{aligned}$$

2a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}v_1(t) &:= x_1'(t) \\v_2(t) &:= x_2'(t)\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}x_1' &= v_1 \\x_2' &= v_2 \\v_1' &= -3 x_1 + x_2 \\v_2' &= 2 x_1 - 2 x_2 + 40 \sin(3 t) \\x_1(0) &= b_1 \\v_1(0) &= b_2 \\x_2(0) &= c_1 \\v_2(0) &= c_2 .\end{aligned}$$

2b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

It is always the case that the natural initial value problems for single or systems of differential equations are equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example.

A special case of this fact is that if you have an IVP for single n^{th} order DE for $x(t)$, it is equivalent to an IVP for a system of n first-order DE's, in which the functions are

$$x_1(t) = x(t), \quad x_2(t) := x'(t), \quad x_3(t) := x''(t), \dots, \quad x_n(t) := x^{(n-1)}(t).$$

For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$

$$x(0) = 4$$

$$x'(0) = -4.$$

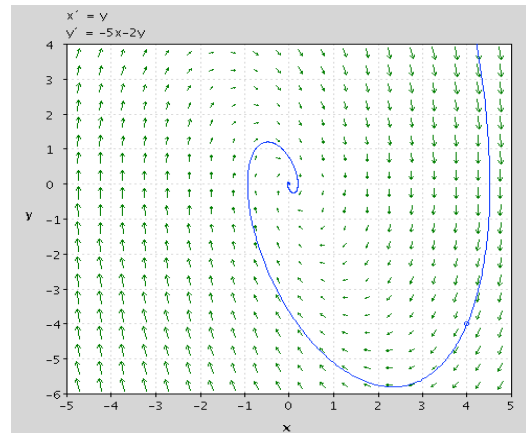
Exercise 3)

3a) Convert this single second order IVP into an equivalent first order system IVP for $x_1(t) := x(t)$ and $x_2(t) := x'(t)$.

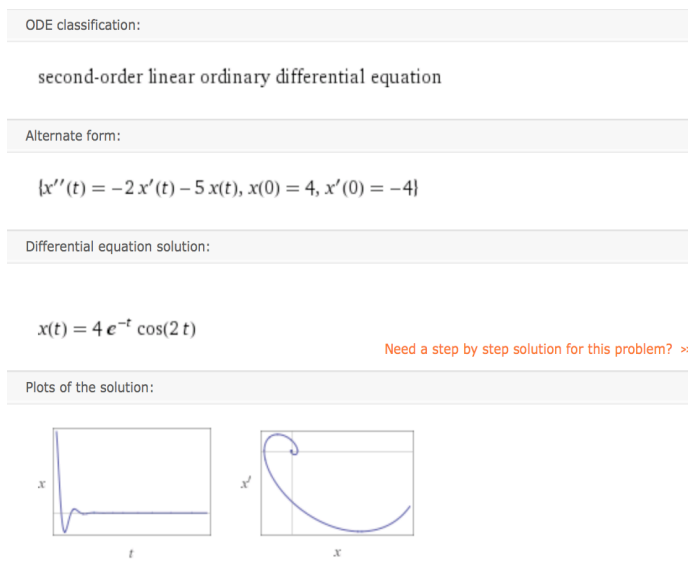
3b) Solve the second order IVP in order to deduce a solution to the first order IVP. Use Chapter 5 methods even though you love Laplace transform more.

3c) How does the Chapter 5 "characteristic polynomial" in 3b compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 3a? hmmm.

3d) Is your analytic solution $[x(t), v(t)]$ in 3b consistent with the parametric curve shown on the next page, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.



If you've been using Wolfram alpha to solve second order differential equations you might have noticed pictures that look just like the one above, even though you might not have thought about them ... look at the plot at the lower right corner of this screenshot!



Theorem 1 For the IVP

$$\begin{aligned}\mathbf{x}'(t) &= \mathbf{F}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

If $\mathbf{F}(t, \mathbf{x})$ is continuous in the t -variable and differentiable in its \mathbf{x} variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\begin{aligned}\mathbf{x}'(t) &= A(t)\mathbf{x}(t) + \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

If the matrix $A(t)$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval I containing t_0 then a solution $\mathbf{x}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commercial numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$\begin{aligned}t_j &= t_0 + hj \\ \mathbf{x}_{j+1} &= \mathbf{x}_j + h \mathbf{F}(t_j, \mathbf{x}_j) .\end{aligned}$$

Remark: These theorems are the true explanation for why the n^{th} -order linear DE IVPs in Chapter 5 always have unique solutions - because each n^{th} - order linear DE IVP is equivalent to an IVP for a first order system of n linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{x}(t)$ differentiable on an interval, the operator

$$L(\underline{x}(t)) := \underline{x}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$\begin{aligned} L(\underline{x}(t) + \underline{z}(t)) &= L(\underline{x}(t)) + L(\underline{z}(t)) \\ L(c \underline{x}(t)) &= c L(\underline{x}(t)) . \end{aligned}$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{f}(t)$$

$\forall t \in I$ is

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$$

where $\underline{x}_p(t)$ is any single particular solution and $\underline{x}_H(t)$ is the general solution to the homogeneous problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{0}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{x}'(t) = A(t)\underline{x}(t) .$$

3.3) For $A(t)_{n \times n}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\mathbf{x}' = A \mathbf{x}$$

is n-dimensional. Here's why:

- Let $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$[W(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)](t_0) := [\mathbf{X}_1(t_0) | \mathbf{X}_2(t_0) | \dots | \mathbf{X}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\mathbf{b} \in \mathbb{R}^n$ the IVP

$$\begin{aligned} \mathbf{x}' &= A \mathbf{x} \\ \mathbf{x}(t_0) &= \mathbf{b} \end{aligned}$$

has solution $\mathbf{x}(t) = c_1 \mathbf{X}_1(t) + c_2 \mathbf{X}_2(t) + \dots + c_n \mathbf{X}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\begin{bmatrix} \mathbf{X}_1(t_0) & \mathbf{X}_2(t_0) & \dots & \mathbf{X}_n(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$, and since each IVP has only one solution, $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\mathbf{b} = \mathbf{0}$) is the one with $\mathbf{c} = \mathbf{0}$. Thus $\mathbf{X}_1(t), \mathbf{X}_2(t), \dots, \mathbf{X}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

7.3 Eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\mathbf{x}' = A \mathbf{x}$$

Here's how: We look for a basis of solutions $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$, where \mathbf{v} is a constant vector. Substituting this form of potential solution into the system of DE's above yields the equation

$$\lambda e^{\lambda t} \mathbf{v} = A e^{\lambda t} \mathbf{v} = e^{\lambda t} A \mathbf{v}.$$

Dividing both sides of this equation by the scalar function $e^{\lambda t}$ gives the condition

$$\lambda \mathbf{v} = A \mathbf{v}.$$

- We get a solution every time \mathbf{v} is an eigenvector of A with eigenvalue λ !
- If A is diagonalizable then there is an \mathbb{R}^n basis of eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and solutions

$$\mathbf{X}_1(t) = e^{\lambda_1 t} \mathbf{v}_1, \mathbf{X}_2(t) = e^{\lambda_2 t} \mathbf{v}_2, \dots, \mathbf{X}_n(t) = e^{\lambda_n t} \mathbf{v}_n$$

which are a basis for the solution space on the interval $I = \mathbb{R}$, because the Wronskian matrix at $t = 0$ is the invertible diagonalizing matrix

$$P = [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n]$$

that we considered in Chapter 6.

- If A has complex number eigenvalues and eigenvectors it may still be diagonalizable over \mathbb{C}^n , and we will still be able to extract a basis of real vector function solutions. If A is not diagonalizable over \mathbb{R}^n or over \mathbb{C}^n the situation is more complicated.

Exercise 4a) Use the method above to find the general homogeneous solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4b) Solve the IVP with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

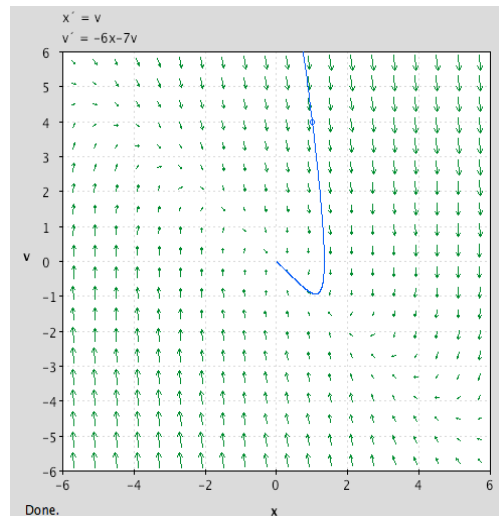
Exercise 5a) What second order overdamped initial value problem is equivalent to the first order system IVP on the previous page. And what is the solution function to this IVP?

5b) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE in 4a, and the Chapter 7 "Wronskian matrix" for the solution to the equivalent first order system?

5c) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Interpret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

5d) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \rightarrow \infty$, and the direction solutions came from (as $t \rightarrow -\infty$)?

<http://math.rice.edu/~dfield/dfpp.html>



7.1-7.3 Summary of what is covered in Monday's and Tuesday's notes:

- Any initial value problem for a differential equation or system of differential equations can be converted into an equivalent initial value problem for a system of first order differential equations.

- There is a "short-time" existence-uniqueness theorem for first order DE initial value problems

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

and solutions can be approximated using Euler or Runge-Kutta type algorithms.

- A special case of the IVP above is the one for a first order linear system of differential equations :

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

If the matrix $A(t)$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval I containing t_0 then a solution $\mathbf{x}(t)$ exists and is unique, on the entire interval.

- The general solution to an inhomogeneous linear system of DEs

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

i.e.

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t)$$

will be of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_H$ where \mathbf{x}_p is a particular solution, and \mathbf{x}_H is the general solution to the homogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) .$$

- For $A(t)_{n \times n}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\mathbf{x}'(t) = A(t) \mathbf{x}(t)$$

i.e.

$$\mathbf{x}'(t) - A(t) \mathbf{x}(t) = \mathbf{0}$$

is n-dimensional.

- For $A_{n \times n}$ a constant matrix, we try to find a basis for the solution space to

$$\mathbf{x}' = A \mathbf{x}$$

consisting of solutions of the form $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$, where \mathbf{v} is an eigenvector of A with eigenvalue λ . We will succeed as long as A is diagonalizable.

Today: We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\mathbf{x}' = A \mathbf{x}$$

that we discussed yesterday, and is in Tuesday's notes. Today we'll consider examples where the eigenvalues and eigenvectors are complex. There is such an example in the homework due Friday.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is \mathbb{C}^n rather than \mathbb{R}^n .

Definition: $\mathbf{v} \in \mathbb{C}^n$ ($\mathbf{v} \neq \mathbf{0}$) is a complex eigenvector of the matrix A , with eigenvalue $\lambda \in \mathbb{C}$ if $A\mathbf{v} = \lambda\mathbf{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. There is a general discussion on the page after this example that we will refer to along the way:

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 3a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

3b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

```

> with(LinearAlgebra) :
> A :=  $\begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$ ;
Eigenvectors(A);

```

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\left[\begin{bmatrix} -\frac{1}{10} + \frac{1}{5} I \\ \frac{1}{10} - \frac{1}{5} I \end{bmatrix}, \begin{bmatrix} 2 I & -2 I \\ 1 & 1 \end{bmatrix} \right]$$

(1)

Notice that Maple writes a capital $I = \sqrt{-1}$.

3c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

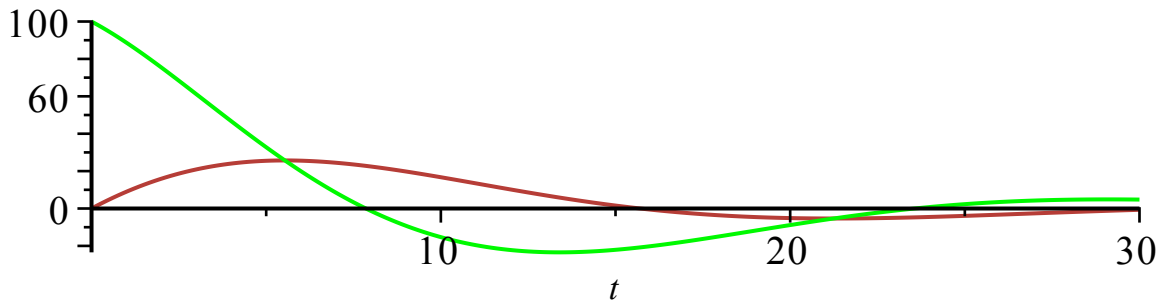
3d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t -> 100 * exp(-.1 * t) * cos(.2 * t) :
  H := t -> 50 * exp(-.1 * t) * sin(.2 * t) :
  plot1 := plot(G(t), t = 0 .. 30, color = green) :
  plot2 := plot(H(t), t = 0 .. 30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

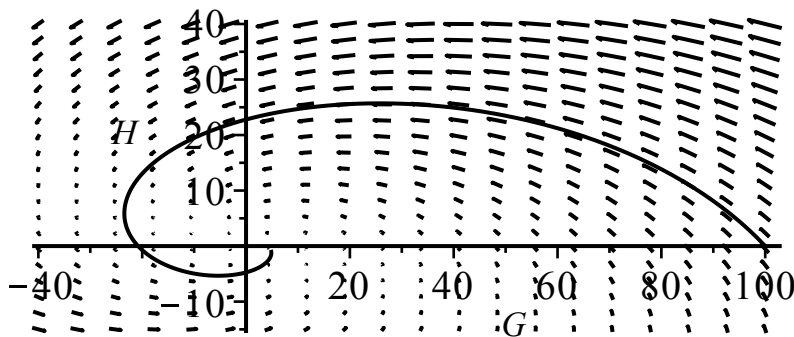
underdamped glucose-insulin interactions



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([- .1 * G - .4 * H, .1 * G - .1 * H], G = -40 .. 100, H = -15 .. 40) :
  soltn := plot([G(t), H(t), t = 0 .. 30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```

Glucose vs Insulin phase portrait



- The example we just worked is linear, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these.

Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\mathbf{x}'(t) = A \mathbf{x}$$

Let A be a real number matrix. Let

$$\lambda = a + b i \in \mathbb{C}$$

$$\mathbf{y} = \boldsymbol{\alpha} + i \boldsymbol{\beta} \in \mathbb{C}^n$$

satisfy $A \mathbf{y} = \lambda \mathbf{y}$, with $a, b \in \mathbb{R}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$.

- Then $\mathbf{z}(t) = e^{\lambda t} \mathbf{y}$ is a complex solution to

$$\mathbf{x}'(t) = A \mathbf{x}$$

because $\mathbf{z}'(t) = \lambda e^{\lambda t} \mathbf{y}$ and this is equal to $A \mathbf{z} = A e^{\lambda t} \mathbf{y} = e^{\lambda t} A \mathbf{y} = e^{\lambda t} \lambda \mathbf{y}$.

- But if we write $\mathbf{z}(t)$ in terms of its real and imaginary parts,

$$\mathbf{z}(t) = \mathbf{x}(t) + i \mathbf{y}(t)$$

then the equality

$$\mathbf{z}'(t) = A \mathbf{z}$$

$$\Rightarrow \mathbf{x}'(t) + i \mathbf{y}'(t) = A(\mathbf{x}(t) + i \mathbf{y}(t)) = A \mathbf{x}(t) + i A \mathbf{y}(t).$$

Equating the real and imaginary parts on each side yields

$$\mathbf{x}'(t) = A \mathbf{x}(t)$$

$$\mathbf{y}'(t) = A \mathbf{y}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A(\boldsymbol{\alpha} + i \boldsymbol{\beta}) = (a + b i)(\boldsymbol{\alpha} + i \boldsymbol{\beta})$ then it is straightforward to check that $A(\boldsymbol{\alpha} - i \boldsymbol{\beta}) = (a - b i)(\boldsymbol{\alpha} - i \boldsymbol{\beta})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i b)t}(\boldsymbol{\alpha} - i \boldsymbol{\beta}) = \mathbf{x}(t) - i \mathbf{y}(t)$$

so yields the same two real solutions (except with a sign change on the second one). Another way to understand how we get the two real solutions is to take the two complex solutions

$$\mathbf{z}(t) = \mathbf{x}(t) + i \mathbf{y}(t)$$

$$\mathbf{w}(t) = \mathbf{x}(t) - i \mathbf{y}(t)$$

and recover $x(t), y(t)$ as linear combinations of these homogeneous solutions:

$$\mathbf{x}(t) = \frac{1}{2} (\mathbf{z}(t) + \mathbf{w}(t))$$

$$\mathbf{y}(t) = \frac{1}{2i} (\mathbf{z}(t) - \mathbf{w}(t)).$$

Summary of this week:

- Any initial value problem for a differential equation or system of differential equations can be converted into an equivalent initial value problem for a system of first order differential equations.

- There is a "short-time" existence-uniqueness theorem for first order DE initial value problems

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

and solutions can be approximated using Euler or Runge-Kutta type algorithms.

- A special case of the IVP above is the one for a first order linear system of differential equations :

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

If the matrix $A(t)$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval I containing t_0 then a solution $\mathbf{x}(t)$ exists and is unique, on the entire interval.

- The general solution to an inhomogeneous linear system of DEs

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

i.e.

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t)$$

will be of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_H$ where \mathbf{x}_p is a particular solution, and \mathbf{x}_H is the general solution to the homogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) .$$

- For $A(t)_{n \times n}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\mathbf{x}'(t) = A(t) \mathbf{x}(t)$$

i.e.

$$\mathbf{x}'(t) - A(t) \mathbf{x}(t) = \mathbf{0}$$

is n-dimensional.

- For $A_{n \times n}$ a constant matrix, we try to find a basis for the solution space to

$$\mathbf{x}' = A \mathbf{x}$$

consisting of solutions of the form $\mathbf{x}(t) = e^{\lambda t} \mathbf{y}$, where \mathbf{y} is an eigenvector of A with eigenvalue λ . We will succeed as long as A is diagonalizable.

Finish 7.3 Applications of first order systems of differential equations

- Here are some of the details from Wednesday's discussion of the **Glucose-insulin model**...

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

- Characteristic polynomial $|A - \lambda I| = (\lambda + .1)^2 + .04$ has roots $\lambda = -.1 \pm .2 i$.
- For the eigenvalue $\lambda = -.1 + .2 i$ we wish to solve the eigenvector system $(A - \lambda I)\mathbf{y} = \mathbf{0}$:

$$\left[\begin{array}{cc|c} -.2 i & -0.4 & 0 \\ 0.1 & -.2 i & 0 \end{array} \right]$$

$$10 R_2 \rightarrow R_1, -5 R_1 \rightarrow R_2:$$

$$\left[\begin{array}{cc|c} 1 & -2 i & 0 \\ i & 2 & 0 \end{array} \right]$$

Although it doesn't look like it, the second row is a multiple of the first row, as it must be:

$$-iR_1 + R_2 \rightarrow R_2:$$

$$\left[\begin{array}{cc|c} 1 & -2 i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

So we may choose the eigenvector $\mathbf{y} = [2 i, 1]^T$. This yields a complex function solution

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{y} = e^{(-.1 + 2 i)t} \begin{bmatrix} 2 i \\ 1 \end{bmatrix}.$$

If we rewrite $\mathbf{z}(t) = \mathbf{x}(t) + i \mathbf{y}(t)$ with $\mathbf{x}(t), \mathbf{y}(t)$ real vector functions, then each of $\mathbf{x}(t), \mathbf{y}(t)$ will be real solutions to the system of differential equations, and will in fact be a basis. (See general discussion at end of Wednesday's notes.) At the end of class we did this decomposition of $\mathbf{z}(t)$ into real and imaginary parts:

$$\begin{aligned} e^{(-.1 + 2 i)t} \begin{bmatrix} 2 i \\ 1 \end{bmatrix} &= e^{-.1 t} (\cos(.2 t) + i \sin(.2 t)) \begin{bmatrix} 2 i \\ 1 \end{bmatrix} \\ &= e^{-.1 t} \begin{bmatrix} (\cos(.2 t) + i \sin(.2 t)) 2 i \\ (\cos(.2 t) + i \sin(.2 t)) 1 \end{bmatrix} = e^{-.1 t} \begin{bmatrix} -2 \sin(.2 t) \\ \cos(.2 t) \end{bmatrix} + i e^{-.1 t} \begin{bmatrix} 2 \cos(.2 t) \\ \cos(.2 t) \end{bmatrix}. \end{aligned}$$

This gives real solutions

$$\mathbf{x}(t) = e^{-.1 t} \begin{bmatrix} -2 \sin(.2 t) \\ \cos(.2 t) \end{bmatrix}, \mathbf{y}(t) = e^{-.1 t} \begin{bmatrix} 2 \cos(.2 t) \\ \sin(.2 t) \end{bmatrix}$$

and general homogeneous solution using real functions:

$$\mathbf{x}_H(t) = c_1 \mathbf{x}(t) + c_2 \mathbf{y}(t) = c_1 e^{-.1 t} \begin{bmatrix} -2 \sin(.2 t) \\ \cos(.2 t) \end{bmatrix} + c_2 e^{-.1 t} \begin{bmatrix} 2 \cos(.2 t) \\ \sin(.2 t) \end{bmatrix}.$$

- The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetaminophen (brand name Tylenol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

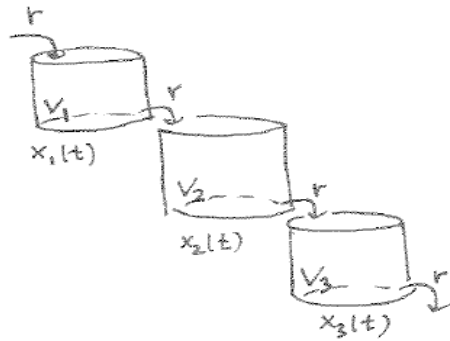
Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

<http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full>

<http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf>

APAP	$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$
NAPQI	$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$
GSH	$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$
Functional Hepatocytes	$\frac{dH}{dt} = rH \left(1 - \frac{H+Z}{H_{max}} \right) - \eta NH$
Damaged Hepatocytes	$\frac{dZ}{dt} = \eta NH - \delta_z Z$
AST	$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$
ALT	$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$
Clotting Factor	$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F \right)$

Example) consider the three component input-output model below:



Exercise 1a) Derive the first order system for the tank cascade above.

1b) In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10 \frac{\text{gal}}{\text{min}}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 422-434...but we'll modify the IVP, and then consider a second case as well.)

1c) Here's the eigenvector data for the matrix in b. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, because otherwise Maple would have given (confusing) floating point answers. Use the eigendata to write down the general solution to the system in b.

```

[> with(LinearAlgebra) :
> A := Matrix(3, 3, [-1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5]);
    Eigenvectors(A);

```

$$A := \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} \\ \frac{1}{5} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{3}{5} \\ -\frac{1}{5} & 0 & -\frac{6}{5} \\ 1 & 1 & 1 \end{bmatrix}$$

(2)

```

[>

```

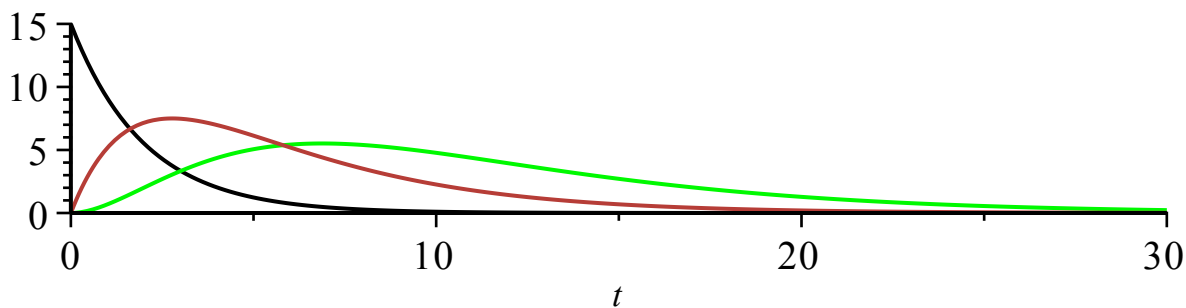
1d) Solve the IVP for this tank cascade, assuming that there are initially 15 lb of salt in the first tank, and no salt in the second and third tanks.

Your answer to d should be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5 e^{-0.5 t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30 e^{-0.25 t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125 e^{-0.2 t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots):
> x1 := t -> 15 * exp(-.5 * t):
  plot1 := plot(x1(t), t = 0..30, color = black):
  x2 := t -> -30 * exp(-.5 * t) + 30 * exp(-.25 * t):
  plot2 := plot(x2(t), t = 0..30, color = brown):
  x3 := t -> 25 * exp(-.5 * t) - 150 * exp(-.25 * t) + 125 * exp(-.2 * t):
  plot3 := plot(x3(t), t = 0..30, color = green):
  display({plot1, plot2, plot3}, title = 'pollutant flushing in tank cascade');
```



Exercise 2) Use the same tank cascade. Only now, assume that there is initially 13 lb salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration $0.5 \frac{\text{lb}}{\text{gal}}$.

2a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.$$

2b) Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{x}_p(t) = \underline{c}.$$

Plug this guess into the inhomogeneous system to deduce \underline{c} . How could your irritating younger sibling have told you this particular solution, without knowing anything at all about linear algebra or differential equations?

2c) Use $\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$ to solve the IVP. Compare your solution to the plots below.

