Math 2250-004
Week 13.
Monday Nov 21: We will use last Friday's notes to discuss impulse forcing and the convolution table entry for Laplace transforms.

Because it is so important it's worth highlighting that for any non-homogeneous forced oscillation problem, the solution function $x(t)$ for the IVP that starts with zero initial conditions can be written as a convolution integral: For the IVP

$$a x'' + b x' + c x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

Then in Laplace land, this equation is equivalent to

$$a s^2 X(s) + b s X(s) + c X(s) = F(s)$$
$$\Rightarrow X(s)\left(a s^2 + b s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s) \cdot \frac{1}{a s^2 + b s + c} = F(s) \cdot W(s)$$

for

$$W(s) := \frac{1}{a s^2 + b s + c}.$$

The inverse Laplace transform $w(t) = \mathcal{L}^{-1}\{W(s))\}(t)$ is called the "weight function" of the given differential equation.

Because of the convolution table entry

$\int_0^t f(\tau)g(t - \tau)\, d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

the solution (for ANY forcing function $f(t)$) is given by

$$x(t) = \int_0^t f(\tau)w(t - \tau)\, d\tau.$$

(With non-zero initial conditions there would be homogeneous solution terms as well. In the case of damping these terms would be transient.)

Tuesday Nov 22

<u>6.1-6.2</u> Eigenvalues and eigenvectors for square matrices.

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear <u>systems of differential equations,</u> in Chapter 7. Such systems of DE's arise naturally in the contexts of

 • coupled input-output models, with several components.
 • coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

<u>Example</u> Consider the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that for the standard basis vectors $\underline{e}_1 = [1, 0]^T, \underline{e}_2 = [0, 1]^T$

$$T(\underline{e}_1) = 3\underline{e}_1$$
$$T(\underline{e}_2) = \underline{e}_2 .$$

The facts that T is linear and that it transforms $\underline{e}_1, \underline{e}_2$ by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(x_1\underline{e}_1 + x_2\underline{e}_2) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2)$$
$$= x_1(3\underline{e}_1) + x_2(1\underline{e}_2) .$$

In other words, T stretches by a factor of 3 in the $\underline{e}_1$ direction, and by a factor of 1 in the $\underline{e}_2$ direction, transforming a square grid in the domain into a parallel rectangular grid in the image:
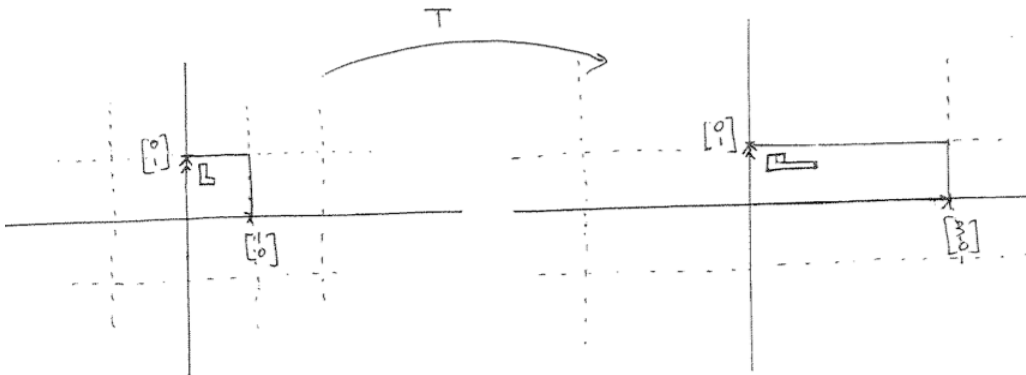
<u>Exercise 1)</u> Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

<u>Exercise 2)</u> And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

<u>Definition:</u> If $A_{n \times n}$ and if $A\underline{v} = \lambda \underline{v}$ for a scalar λ and a vector $\underline{v} \neq \underline{0}$ then $\underline{v}$ is called an <u>eigenvector of A</u>, and λ is called the <u>eigenvalue</u> of $\underline{v}$. (In some texts the words <u>characteristic vector</u> and <u>characteristic value</u> are used as synonyms for these words.)

• In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

<u>Exercise 3)</u> Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors $\underline{x}$ and computing $A\,\underline{x}$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

<u>How to find eigenvalues and eigenvectors (including eigenspaces) systematically</u>:

If

$$A\,\underline{v} = \lambda\,\underline{v}$$
$$\Leftrightarrow A\,\underline{v} - \boldsymbol{\lambda}\,\underline{v} = \underline{\mathbf{0}}$$
$$\Leftrightarrow A\,\underline{v} - \lambda\,I\,\underline{v} = \underline{\mathbf{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\,I)\underline{v} = \underline{\mathbf{0}}\,.$$

As we know, this last equation can have non-zero solutions $\underline{v}$ if and only if the matrix $(A - \lambda\,I)$ is not invertible, i.e.

$$\Leftrightarrow det(A - \lambda\,I) = 0\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\,I)\,.$$

If $A_{n\,\times\,n}$ then $p(\lambda)$ will be degree n. This polynomial is called the <u>characteristic polynomial</u> of the matrix A.

- λ_j can be an eigenvalue for some non-zero eigenvector $\underline{v}$ if and only if it's a root of the characteristic polynomial, i.e. $p(\lambda_j) = 0$. For each such root, the homogeneous solution space of vectors $\underline{v}$ solving

$$(A - \lambda_j\,I)\underline{v} = \underline{\mathbf{0}}$$

will be eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\,I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

<u>Notation</u>: The subspace of eigenvectors for eigenvalue λ_j is called the $\underline{\lambda_j\text{ eigenspace}}$, and denoted by E_{λ_j}.

(We include the zero vector in E_{λ_j}.) The basis of eigenvectors is called an <u>eigenbasis</u> for E_{λ_j}.

Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues.
(ii) for each eigenvalue, find bases for the corresponding eigenspaces.
(iii) Can you describe the transformation $T(\underline{x}) = B\underline{x}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

Your solution will be related to the output below:

```
> with(LinearAlgebra) :
  B := Matrix(3, 3, [4, -2, 1, 2, 0, 1, 2, -2, 3]);
  Eigenvectors(B);
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$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -\dfrac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad (1)$$

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In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{x}) = A\,\underline{x}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 7: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Math 2250-004
Wed Nov 23
6.1-6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability of matrices.

Recall from yesterday,

<u>Definition</u>: If $A_{n \times n}$ and if $A\,\underline{v} = \lambda\,\underline{v}$ for some scalar λ and vector $\underline{v} \neq \underline{0}$ then $\underline{v}$ is called an <u>eigenvector</u> of A, and λ is called the <u>eigenvalue</u> of $\underline{v}$ (and an eigenvalue of A).

*   For general matrices, the eigenvector equation $A\,\underline{v} = \lambda\,\underline{v}$ can be rewritten as
$$(A - \lambda\,I)\underline{v} = \underline{0} \ .$$
The only way such an equation can hold for $\underline{v} \neq \underline{0}$ is if the matrix $(A - \lambda\,I)$ does not reduce to the identity matrix. In other words - $det(A - \lambda\,I)$ must equal zero. Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the <u>characteristic polynomial</u>
$$p(\lambda) = det(A - \lambda\,I) \ .$$
*   So, the first step in finding eigenvectors for A is actually to find the eigenvalues - by finding the characteristic polynomial and its roots λ_j .

*   For each root λ_j the matrix $A - \lambda_j\,I$ will not reduce to the identity, and the solution space to
$$\left(A - \lambda_j\,I\right)\underline{v} = \underline{0}$$
will be at least one-dimensional, and have a basis of one or more eigenvectors. Find such a basis for this λ_j <u>eigenspace</u> $E_{\lambda = \lambda_j}$ by reducing the homogeneous matrix equation
$$\left(A - \lambda_j\,I\right)\underline{v} = \underline{0} \ ,$$
backsolving, and extracting a basis.

<u>Exercise 1a)</u> Do this exercise from yesterday's notes. The matrix for which we want to find eigenvalues and eigenvectors (eigenspace bases) is
$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \ .$$

<u>step 1)</u>

$$|B - \lambda\,I| = \begin{bmatrix} 4 - \lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3 - \lambda \end{bmatrix} =$$

<u>step 2)</u> Find bases for the eigenspaces. Talk about how you are able to use the fact that homogeneous solutions to matrix equations correspond to column dependencies, in order to read off eigenspace bases more quickly than by backsolving.... ask questions if this is not clear, because you'll be computing a lot of eigenvalues and eigenvectors in the next chapters and you'll save a lot of time if you get comfortable with this shortcut. (Of course, for most larger matrices one just uses technology for eigenvalue/eigenvector computation.)

<u>1b)</u> Notice that you can construct a basis for $\mathbb{R}^3$ by combining your eigenspace bases from <u>1a</u>. Use this fact to describe the geometry of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$T(\underline{x}) = B\,\underline{x}.$$

Your algebraic work in <u>1a</u> is related to the output below:

> *with(LinearAlgebra)* :
  $B := Matrix(3, 3, [4, -2, 1, 2, 0, 1, 2, -2, 3])$;
  Eigenvectors(B);

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{c} 3 \\ 2 \\ 2 \end{array}\right], \begin{bmatrix} 1 & -\dfrac{1}{2} & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(2)

Exercise 2) If your matrix A is diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from yesterday, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a 3×3 diagonal matrix, so that in the future you can just read of the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

step 1) Find the roots of the characteristic polynomial $det(A - \lambda I)$.

step 2) Find the eigenspace bases, assuming the values of a_{11}, a_{22}, a_{33} are distinct (all different). What if $a_{11} = a_{22}$ but these values do not equal a_{33} ?

In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{x}) = A \, \underline{x}$$

almost as well as if A is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 1, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda = 2}$ basis vectors in the first two columns, and the $E_{\lambda = 3}$ basis vector in the third column:

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B P = P D ,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P). Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P D P^{-1} \text{ and } P^{-1} B P = D.$$

Exercise 3) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

<u>Definition:</u> Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{v}_1, \underline{v}_2, ..., \underline{v}_n$ consisting of eigenvectors of A, then A is called <u>diagonalizable.</u> This is precisely why:

Write $A \underline{v}_j = \lambda_j \underline{v}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix
$$P = \left[\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right].$$
Then, using the various ways of understanding matrix multiplication, we see

$$A P = A\left[\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right] = \left[\lambda_1 \underline{v}_1 \middle| \lambda_2 \underline{v}_2 \middle| \cdots \middle| \lambda_n \underline{v}_n \right]$$

$$= \left[\underline{v}_1 \middle| \underline{v}_2 \middle| \cdots \middle| \underline{v}_n \right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

$$A P = P D$$
$$A = P D P^{-1}$$
$$P^{-1} A P = D .$$

Unfortunately, not all matrices are diagonalizable:
<u>Exercise 4)</u> Show that
$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is <u>not</u> diagonalizable.

<u>Facts about diagonalizability</u> (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda = \lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called <u>defective.</u>
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ each eigenspace is forced to be 1-dimensional since $k_1 + k_2 + \ldots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

<u>Exercise 5)</u> How do the examples from today and yesterday compare with the general facts about diagonalizability?