Math 2250-004 Week 12 continue 10.1-10.3; also cover parts of 10.4-10.5, EP 7.6
Mon Nov 14

10.1-10.3 Laplace transform, and application to DE IVPs, especially those in Chapter 5. Today we'll continue (from last Wednesday) to fill in the Laplace transform table (on page 2), and to use the table entries to solve linear differential equations.

Exercise 1) (to review) Use the table to compute

1a) $\mathcal{L}\{4 - 5\cos(3t) + 2e^{-4t}\sin(12t)\}(s)$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{1}{s^2 + 2s + 5}\right\}(t)$.

Exercise 2) (to review) Use Laplace transforms to solve the IVP we didn't get to lastWednesday, for an underdamped, unforced oscillator DE. Compare to Chapter 5 method.

$$x''(t) + 6x'(t) + 34x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	
t^2	$\frac{2}{s^3}$	
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(\alpha))$	☐
$\cos(k\,t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$	☐
$\sin(k\,t)$	$\frac{k}{s^2+k^2}$ $(s > 0)$	☐
$\cosh(k\,t)$	$\frac{s}{s^2-k^2}$ $(s > k)$	
$\sinh(k\,t)$	$\frac{k}{s^2-k^2}$ $(s > k)$	
$e^{a\,t}\cos(k\,t)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{a\,t}\sin(k\,t)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$	☐
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	
$t\,f(t)$	$-F'(s)$	
$t^2\,f(t)$	$F''(s)$	
$t^n\,f(t),\ n \in \mathbb{Z}$	$(-1)^n\,F^{(n)}(s)$	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	
$t\,\cos(k\,t)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	
$\frac{1}{2\,k}\,t\,\sin(k\,t)$	$\frac{s}{(s^2+k^2)^2}$	

$\frac{1}{2k^3}(\sin(kt) - kt\cos(kt))$	$\frac{1}{(s^2+k^2)^2}$	
$e^{at}f(t)$ $t\,e^{at}$ $t^n\,e^{at},\ n \in \mathbb{Z}$	$F(s-a)$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	
more after exam!		

Laplace transform table

work down the table ...

3a) $\mathcal{L}\{cosh(k\,t)\}(s) = \frac{s}{s^2 - k^2}$

3b) $\mathcal{L}\{\sinh(k\,t)\}(s) = \frac{k}{s^2 - k^2}$.

Exercise 4) Recall we used integration by parts on Wednesday to derive

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)\ .$$

Use that identity to show

a) $\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s\,f(0) - f'(0)$,

b) $\mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - s\,f'(0) - f''(0)$,

c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ \ldots\ - f^{(n-1)}(0),\ n \in \mathbb{N}$.

d) $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$.

These are the identities that make Laplace transform work so well for initial value problems such as we studied in Chapter 5.

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 4d.

b) using partial fractions.

Exercise 6) Show $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4.

Math 2250-004
Tues Nov 15

10.2-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.
Today we'll continue to fill in the Laplace transform table, and to use the table entries to solve linear differential equations. One focus today will be to review partial fractions, since the table entries are set up precisely to show the inverse Laplace transforms of the components of partial fraction decompositions.

Exercise 1) Check why this table entry is true - notice that it generalizes how the Laplace transforms of $\cos(k\,t)$, $\sin(k\,t)$ are related to those of $e^{a\,t}\cos(k\,t)$, $e^{a\,t}\sin(k\,t)$:

$e^{a\,t}f(t)$	$F(s-a)$

Exercise 2) Verify the table entry

$t^n,\ n\in\mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$

by applying one of the results from yesterday:

$$\mathcal{L}\{f^{(n)}(t)\}(s)=s^n\mathcal{L}\{f(t)\}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots f^{(n-1)}(0).$$

Exercise 3) Combine 1,2, to get

$t^n e^{a\,t}$	$\dfrac{n!}{(s-a)^{n+1}}$

A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$t\,f(t)$	$-F'(s)$

Here's how we get it:

$$F(s)=\mathcal{L}\{f(t)\}(s):=\int_0^\infty f(t)e^{-s\,t}\,dt$$

$$\Rightarrow\ \frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty f(t)e^{-s\,t}\,dt=\int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt\,.$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt=\int_0^\infty f(t)(-t\,)e^{-s\,t}\,dt=-\mathcal{L}\{t\,f(t)\}(s)\qquad\Box.$$

For resonance and other applications ...

Exercise 4) Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$ or Euler's formula and $\mathcal{L}\{t e^{a t}\}(s) = \frac{1}{(s-a)^2}$ to show

a) $\mathcal{L}\{t\cos(k t)\}(s) = \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$

b) $\mathcal{L}\left\{\frac{1}{2k}\, t\sin(k t)\right\}(s) = \frac{s}{\left(s^2 + k^2\right)^2}$

c) Then use a and the identity

$$\frac{1}{\left(s^2 + k^2\right)^2} = \frac{1}{2k^2}\left(\frac{s^2 + k^2}{\left(s^2 + k^2\right)^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2 + k^2\right)^2}\right\}(t) = \frac{1}{2k^2}\left(\frac{1}{k}\sin(k t) - t\cos(k t)\right).$$

Notice how the Laplace transform table is set up to use partial fraction decompositions. And be amazed at how it lets you quickly deduce the solutions to important DE IVPs, like this resonance problem:

<u>Exercise 5a)</u> Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin(\omega_0\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0\,.$$

what phenomenon do the solutions to this IVP exhibit?

<u>5b)</u> Use Laplace transforms to solve the general undamped forced oscillation problem, when $\omega \neq \omega_0$:

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin(\omega\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{M\,t}$	$F(s) := \int_0^\infty f(t)e^{-s\,t}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s}\quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐
$e^{\alpha\,t}$	$\frac{1}{s-\alpha}\quad (s > \Re(a))$	☐
$\cos(k\,t)$ $\sin(k\,t)$ $\cosh(k\,t)$ $\sinh(k\,t)$ $e^{a\,t}\cos(k\,t)$ $e^{a\,t}\sin(k\,t)$ $e^{a\,t}f(t)$	$\frac{s}{s^2+k^2}\quad (s > 0)$ $\frac{k}{s^2+k^2}\quad (s > 0)$ $\frac{s}{s^2-k^2}\quad (s > k)$ $\frac{k}{s^2-k^2}\quad (s > k)$ $\frac{(s-a)}{(s-a)^2+k^2}\quad (s > a)$ $\frac{k}{(s-a)^2+k^2}\quad (s > a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☐ ☐ ☐ ☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	
$t\cos(k\,t)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	

$\frac{1}{2k} t \sin(k t)$	$\frac{s}{(s^2 + k^2)^2}$	
$\frac{1}{2k^3}(\sin(k t) - k t \cos(k t))$	$\frac{1}{(s^2 + k^2)^2}$	
$t e^{a t}$	$\frac{1}{(s - a)^2}$	
$t^n e^{a t},\ n \in \mathbb{Z}$	$\frac{n!}{(s - a)^{n+1}}$	
more after the midterm!		

Laplace transform table

Exercise 6) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4x(t) = 8te^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

Exercise 7a) What is the form of the partial fractions decomposition for

$$X(s) = \frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\,\left((s+1)^2 + 4\right)\,\left(s^2 + 4\right)}\ .$$

7b) Have Maple compute the precise partial fractions decomposition.
7c) What is $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$?
7d) Have Maple compute the inverse Laplace transform directly.

> $convert\left(\frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\,\left((s+1)^2 + 4\right)\,\left(s^2 + 4\right)}, parfrac\right);$

> $with(inttrans);$

[*addtable*, *fourier*, *fouriercos*, *fouriersin*, *hankel*, *hilbert*, *invfourier*, *invhilbert*, *invlaplace*, *invmellin*, *laplace*, *mellin*, *savetable*] **(1)**

> $invlaplace\left(\frac{-356 + 45\cdot s - 100\cdot s^2 - 4\cdot s^5 - 9\cdot s^4 + 39\cdot s^3 + s^6}{(s-3)^3\cdot\left((s+1)^2 + 4\right)\cdot\left(s^2 + 4\right)}, s, t\right);$

>

Math 2250-4
Wed Nov 16

10.4-10.5

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle. We will continue this discussion on Friday, with a few more table entries including "the delta (impulse) function".

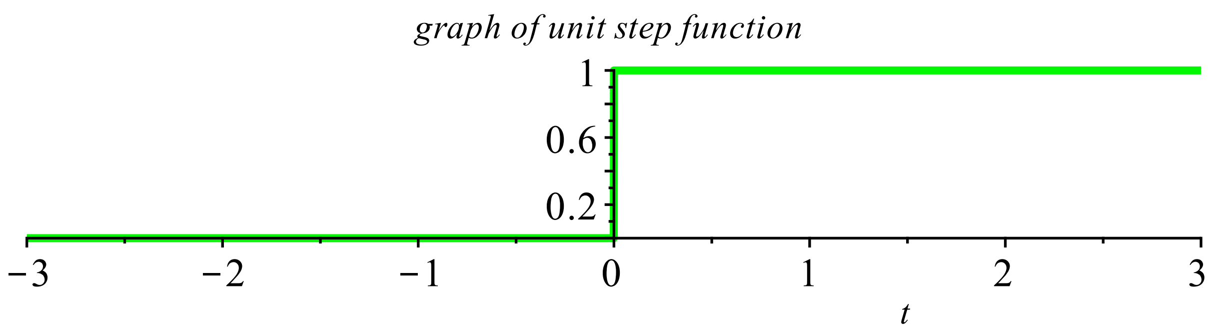
$f(t)$ with $\|f(t)\| \le C e^{Mt}$	$F(s) := \int_0^\infty f(t) e^{-st}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-as} F(s)$	more complicated on/off
$\int_0^t f(t-\tau) f(\tau)\, d\tau$	$F(s) G(s)$	"convolution" for inverting products of Laplace transforms

The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

Its graph is shown below. Notice that this function is called the "Heaviside" function in Maple, after the person who popularized it (among a lot of other accomplishments) and not because it's heavy on one side.
http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots) :
  plot(Heaviside(t), t=-3 ..3, color = green, title = `graph of unit step function`);
```



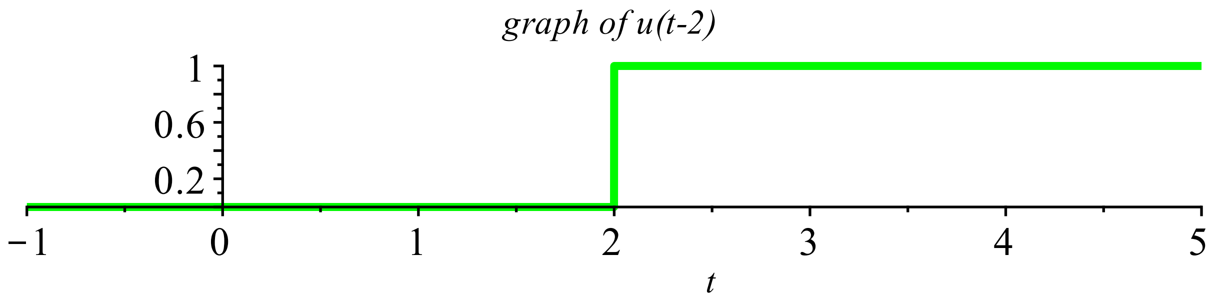
Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0; \textit{ i.e. } t < a \\ 1, & t-a \geq 0; \textit{ i.e. } t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:

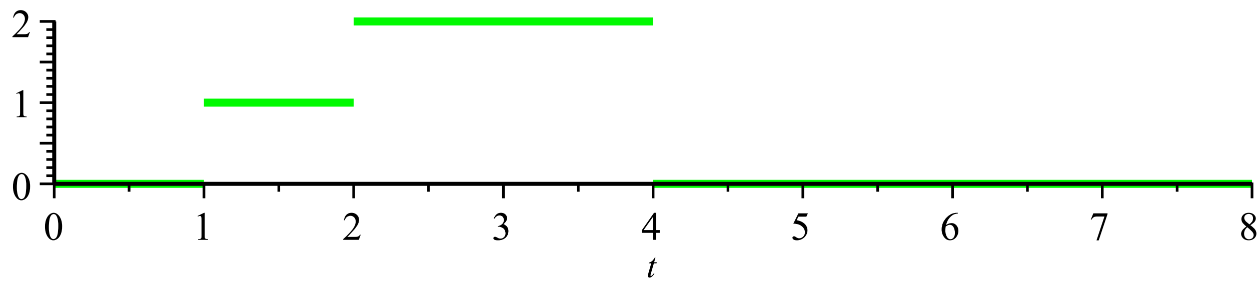
```
> plot(Heaviside(t - 2), t=-1..5, color = green, title = `graph of u(t-2)`);
```



Exercise 1) Verify the table entries

$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)



> $plot(\text{Heaviside}(t-1) + \text{Heaviside}(t-2) - 2\cdot\text{Heaviside}(t-4), t = 0..8, color = green);$
> $with(inttrans):$
> $laplace(\text{Heaviside}(t-1) + \text{Heaviside}(t-2) - 2\cdot\text{Heaviside}(t-4), t, s);$

Setup: an under-employed mathematician/engineer/scientist (your choice)

likes to take his/her child to the swings ...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$

$$\dashrightarrow x'' + g\frac{x}{L} = 0$$

$$\dashrightarrow m x'' + \frac{mg}{L} x = F_0 \cos\omega t \quad \leftarrow \text{parent forcing (?!)}$$

$x(t) = L\sin\theta(t)$
$\doteq L\theta$ for small θ

so $x'' \doteq L\theta''$

$$\dashrightarrow x'' + \frac{g}{L} x = \frac{F_0}{m}\cos\omega t$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

:)

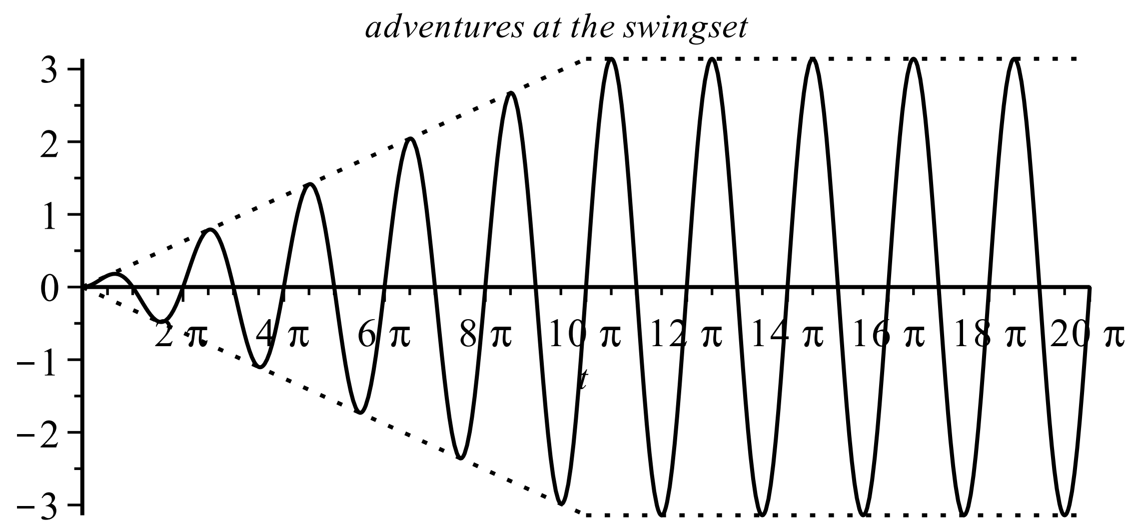
<u>Exercise 3a)</u> Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\left(1 - u(t - 10\pi)\right)$$
$$x(0) = 0$$
$$x'(0) = 0$$

<u>3b)</u> Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 5:

step 1) solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

step 2) Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^{\infty} f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(kt)$	$\frac{s}{s^2+k^2}$ $(s > 0)$	☐
$\sin(kt)$	$\frac{k}{s^2+k^2}$ $(s > 0)$	☐
$\cosh(kt)$	$\frac{s}{s^2-k^2}$ $(s > k)$	☐
		☐
$\sinh(kt)$	$\frac{k}{s^2-k^2}$ $(s > k)$	☐
$e^{at}\cos(kt)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at} f(t)$	$F(s-a)$	☐
$u(t-a)$	$\frac{e^{-as}}{s}$	
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	
$\delta(t-a)$	e^{-as}	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐

$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^\infty F(\sigma)\, d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\frac{1}{2\,k}\, t\sin(k\,t)$ $\frac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\cos(k\,t))$ $t\, e^{a\,t}$ $t^n\, e^{a\,t},\ n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1 - e^{-ps}} \int_0^p f(t) e^{-s\,t}\, dt$	

Laplace transform table

Math 2250-4
Fri Nov 18
10.5, EP7.6
Today we finish discussing Laplace transform techniques:

- Impulse forcing ("delta functions")...today's notes.
- Convolution formulas to solve any inhomogeneous constant coefficient linear DE, with applications to interesting forced oscillation problems...today's notes.

Laplace table entries for today:

$f(t)$ with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-a s}$	unit impulse/delta "function"
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a + \varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\, dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\, v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\, v'(t)\, dt = \int_a^{a+\varepsilon} f(t)\, dt = p$$

$$\Rightarrow m\, v(t) \Big]_{t=a}^{a+\varepsilon} = p\, .$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\, d_{a,\varepsilon}(t)$$

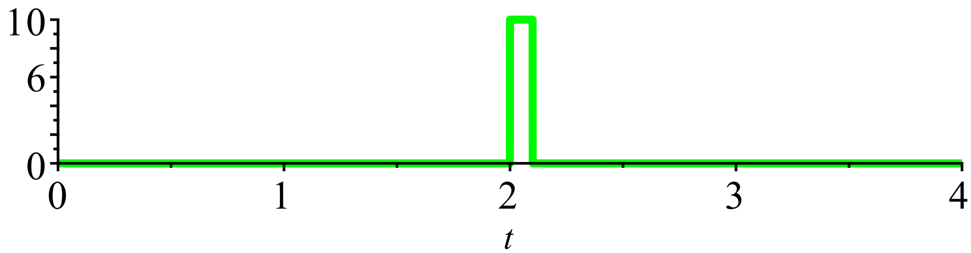
where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a + \varepsilon \\ 0, & t \ge a + \varepsilon \end{cases} .$$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = 1 .$$

Here's a graph of $d_{2,\,.1}(t)$, for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) .$$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-a s} \lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

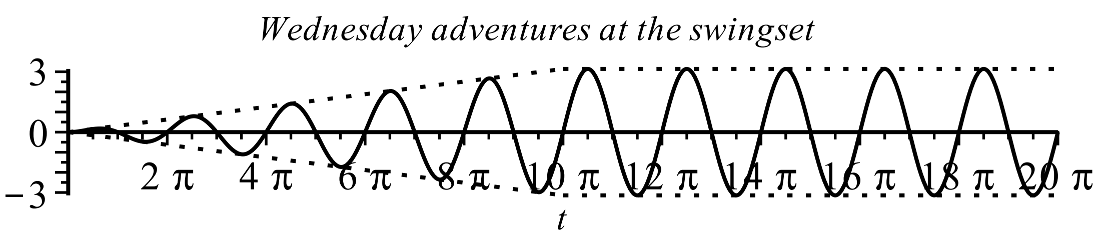
The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 1) Revisit the swing from Wednesday's notes and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi\left[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)\right]$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

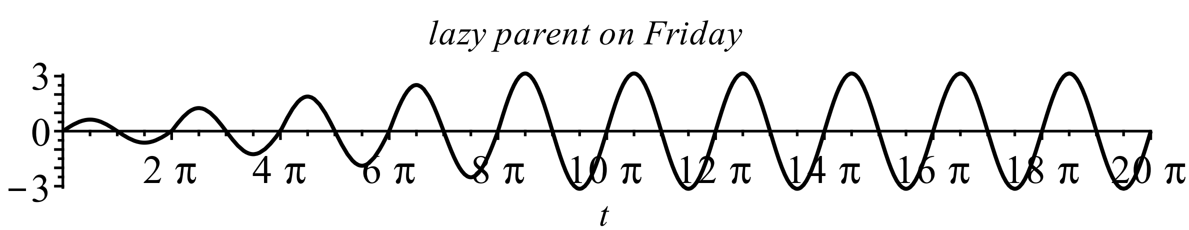
```
> with(plots):
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black):
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black):
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2):
  plot4 := plot(−Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2):
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2):
  plot6 := plot(−.1·t, t = 0..10·Pi, color = black, linestyle = 2):
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```




```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

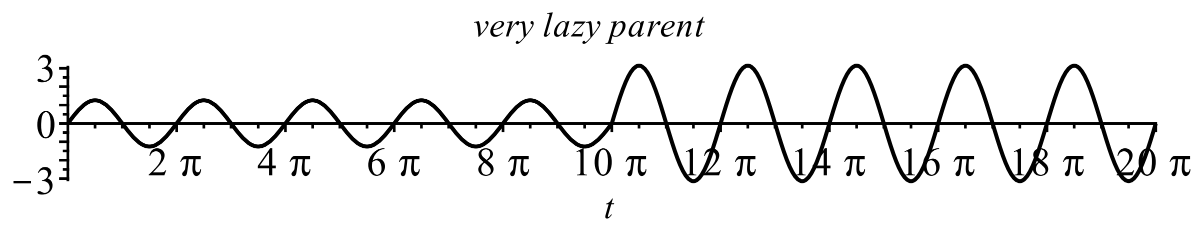
```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0..4):
> plot(f(t), t = 0..20·Pi, color = black, title = `lazy parent on Friday`);
```




```
>
```

Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

```
> g := t→.2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)):
> plot(g(t), t = 0..20·Pi, color = black, title = `very lazy parent`);
```



>

<u>Convolutions and solutions to non-homogeneous physical oscillation problems</u> (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2 X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} := F(s)W(s)\,.$$

Because of the convolution table entry

$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

the solution is given by

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau)\,f(t-\tau)\,d\tau\,.$$

where $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$. The function $w(t)$ is called the "weight function" of the differential equation, because the solution $x(t)$ is some sort of weighted average of the the forces f between times 0 and t, where the weighting factors are given by w in some sort of convoluted way.

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

The mathematical justification for the general convolution table entry is at the end of today's notes, for those who have studied iterated double integrals and who wish to understand it.

Exercise 2. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$x''(t) + x(t) = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

a) Find the weight function $w(t)$.
b) Write down the solution formula for $x(t)$ as a convolution integral.
c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$, and verify that the convolution formula reproduces the answer we would've gotten from the table entry

$\frac{t}{2k}\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$

```
> ∫_0^t sin(τ) cos(t − τ) dτ;
  ∫_0^t cos(τ) sin(t − τ) dτ; #convolution is commutative
>
```

$$\int_0^t \sin(\tau)\cos(t-\tau)\,d\tau;$$
$$\int_0^t \cos(\tau)\sin(t-\tau)\,d\tau; \quad \textit{\#convolution is commutative}$$

d) Then play the resonance game on the following pages with new periodic forcing functions ...

We worked out that the solution to our DE IVP will be

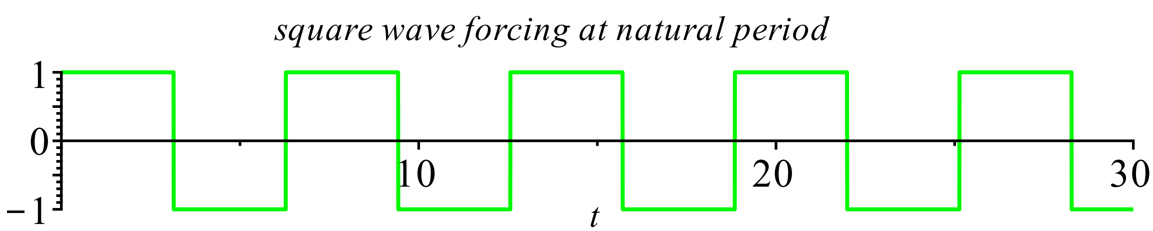
$$x(t) = \int_0^t \sin(\tau) f(t-\tau)\, d\tau$$

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

Example 1) A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$) .

```
> with(plots) :
```

```
> f1 := t→-1 + 2·( Σ_{n=0}^{10} (-1)^n·Heaviside(t - n·Pi) ) :
  plot1a := plot(f1(t), t = 0 ..30, color = green) :
  display(plot1a, title = `square wave forcing at natural period`);
```

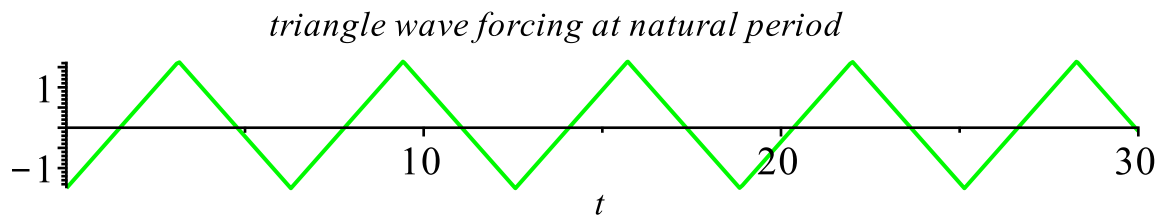


1) What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

```
> x1 := t→∫_0^t sin(τ)·f1(t - τ) dτ :
  plot1b := plot(x1(t), t = 0 ..30, color = black) :
  display({plot1a, plot1b}, title = `resonance response ?`);
```

Example 2) A triangle wave forcing function, same period

```
> f2 := t -> ∫_0^t f1(s) ds - 1.5 :   # this antiderivative of square wave should be triangle wave
  plot2a := plot(f2(t), t = 0..30, color = green) :
  display(plot2a, title = `triangle wave forcing at natural period`);
```

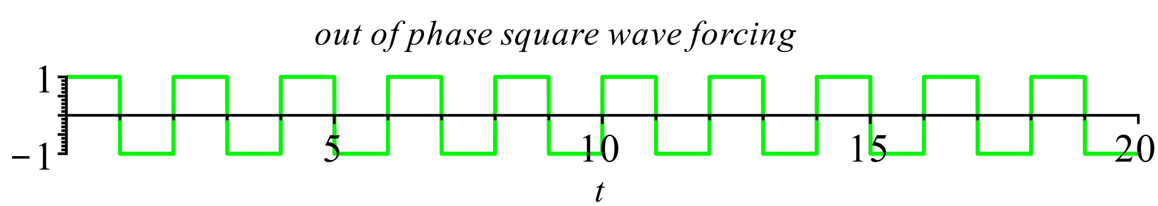


2) Resonance?

```
> x2 := t -> ∫_0^t sin(τ)·f2(t - τ) dτ :
  plot2b := plot(x2(t), t = 0..30, color = black) :
  display({plot2a, plot2b}, title = `resonance response ?`);
```

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

```
> f3 := t -> -1 + 2·Sum((-1)^n·Heaviside(t - n), n = 0..20) :
  plot3a := plot(f3(t), t = 0..20, color = green) :
  display(plot3a, title = `out of phase square wave forcing`);
```

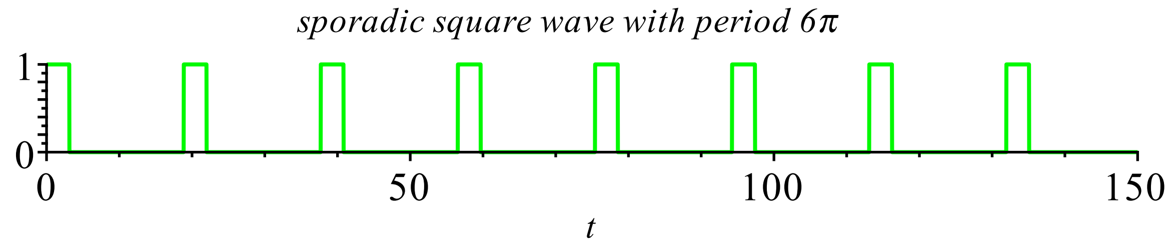


3) Resonance?

```
> x3 := t -> ∫_0^t sin(τ)·f3(t - τ) dτ :
  plot3b := plot(x3(t), t = 0..20, color = black) :
  display({plot3a, plot3b}, title = `resonance response ?`);
```

Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\,\pi$.

> $f4 := t \rightarrow \sum_{n=0}^{10} (\mathrm{Heaviside}(t - 6 \cdot n \cdot \pi) - \mathrm{Heaviside}(t - (6 \cdot n + 1) \cdot \pi))$:
$plot4a := plot(f4(t), t = 0 ..150, color = green)$:
$display(plot4a, title = sporadic\ square\ wave\ with\ period\ 6\pi)$;



4) Resonance?

> $x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t - \tau)\, \mathrm{d}\tau$:
$plot4b := plot(x4(t), t = 0 ..150, color = black)$:
$display(\{plot4a, plot4b\}, title = `resonance\ response\ ?`)$;

>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

<u>**Precise Answer:**</u> It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\{1, \cos(\omega\, t), \sin(\omega\, t), \cos(2\,\omega\, t), \sin(2\,\omega\, t), \cos(3\,\omega\, t), \sin(3\,\omega\, t), \ldots\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0}$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

<u>Exercise 3)</u> The natural period of the following DE is (still) $T_0 = 2\,\pi$. Notice that the period of the first forcing function below is $T = 6\,\pi$ and that the period of the second one is $T = T_0 = 2\,\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas.

<u>a)</u>

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

<u>b)</u>

$$x''(t) + x(t) = \cos(2\,t) - 3\,\sin(3\,t).$$

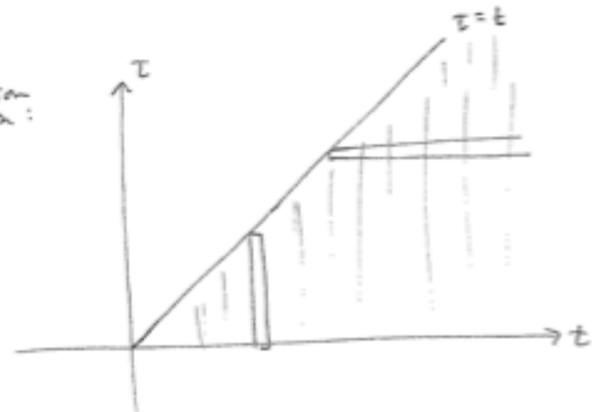
proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau)\, d\tau\, dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\; e^{-s(t-\tau)} g(t-\tau)\, dt\, d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt\right] d\tau$$

$\tilde{t} = t - \tau$

$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\, d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= G(s)\, F(s) \qquad !!$$