Math 2250-004

Week 9 concepts and homework, due October 28.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

Carry over problems from week 8:

5.1: Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.

1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 1, y'(1) = 7$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

w8.3a) In 5.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where the constant r is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of r for which $y(x)$ will satisfy the DE. (See Theorem 5 in the text.)

w8.3b) In 5.1.10 above, the text tells you that $y_1(x) = e^{5x}, y_2(x) = x e^{5x}$ are two independent solutions to the second order homogeneous differential equation $y'' - 10y' + 25y = 0$. Follow the procedure in part a of trying for solutions of the form $y(x) = e^{rx}$, and then use the repeated roots Theorem 6 in the text, to recover these two solutions.

5.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.

1, 2, 5, **8**, 11,13, **16,** 21, **25,** 26

New problems for this week:

5.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

5.3: 3, 9, 11, 23, 27.

w9.1) Do the following problems for homogeneous linear differential equations by hand. They are testing your ability to use the algorithm for finding bases for the solution spaces, based on the characteristic polynomial. Check your work with Maple (or other software). In Maple you will want to use the "dsolve" command to check differential equation solutions, and may want to use the "factor" command to check your factorizations of the characteristic polynomials. Hand in a printout of your computer verifications for the differential equation solutions, along with your written work.

a) Find the general solution to the differential equation for $y(x)$

$$y^{(3)} - 5\,y'' + 3\,y' + 9\,y = 0.$$

Hint: Find a root r_1 of the cubic characteristic polynomial, then divide it by $(r - r_1)$ to get a quotient quadratic polynomial.

b) Find the general solution to the differential equation for $x(t)$

$$x'' + 4\,x' + 29\,x = 0\,.$$

Hint: completing the square works well here - probably better than the quadratic formula.

c) Solve the initial value problem for the differential equation in b, with $x(0) = 0, x'(0) = 10$.

d) Find the general solution to the differential equation for $y(x)$

$$y^{(4)} - 8\,y' = 0.$$

e) Find the general solution to the differential equation for $y(x)$

$$y^{(5)} + 6\,y^{(3)} + 9\,y' = 0.$$

w9.2) Euler's formula

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)$$

is extremely useful in higher mathematics/science/engineering. In class we discuss how this definition is motivated by Taylor series. Amazingly, the rule of exponents is true for such expressions. In other words

$$e^{i\,\alpha + i\,\beta} = e^{i\,\alpha}\,e^{i\,\beta}\,.$$

Check this identity by rewriting the left side as $e^{i\,(\alpha + \beta)}$ and then using Euler's formula to expand both sides. You'll notice that the identity is true because of the addition angle formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (And so this gives a good way to recover the trig. identities if you happen to forget them.)

w9.3) We explore why the general algorithm for finding a basis of solutions for constant coefficient homogeneous linear differential equations yields independent functions, for an illustrative example.

a) What 5^{th} order constant coefficient homogeneous linear DE has characteristic polynomial

$$p(r) = (r-3)^3\left(r^2+16\right) \ ?$$

b) According to our general algorithm, write down a basis for the 5-dimensional solution space to the DE in part a.

c) Prove that the five functions you wrote down in part b actually are linearly independent (so automatically span, so a basis). Hint: Use a combination of the techniques we discussed on Friday February 28, rather than attempting a Wronskian computation. Notice that the solutions exist on the entire real line $-\infty < x < \infty$.

In the next exercise you will continue to explore the reasoning that leads to the "magic" general algorithm for solving constant coefficient homogeneous linear DEs in the case that the characteristic polynomial has repeated roots. These ideas are also discussed in section 5.3, and it may help you to read the book's discussion. Here's a summary: Begin with the derivative operator D defined by $\mathrm{D}(y) := y'$. Then the second derivative operator is $\mathrm{D}^2(y) := \mathrm{D}\circ\mathrm{D}(y)$, etc. So for $y = y(x)$ and

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

we may write

$$L = \mathrm{D}^n + a_{n-1}\mathrm{D}^{n-1} + \ldots + a_1\mathrm{D} + a_0 I$$

where I is the identity operator, $I(y) := y$. In other words, we are writing the operator L as a linear combination of various derivative operators. It turns out that any factorization of the characteristic polynomial $p(r)$ leads to a corresponding factorization of L. For example, consider the second order case with real roots,

$$L = \mathrm{D}^2 + a_1\mathrm{D} + a_0 I.$$

If $p(r)$ factors with real roots,

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$$

Then L factors the same way, as a composition of first order operators:

$$L = \mathrm{D}^2 - (r_1 + r_2)\mathrm{D} + r_1 r_2 I = (\mathrm{D} - r_1 I)\circ(\mathrm{D} - r_2 I).$$

(To check this is to work from right to left above:

$$\begin{aligned}(\mathrm{D} - r_1 I)\circ(\mathrm{D} - r_2 I) &= \mathrm{D}\circ(\mathrm{D} - r_2 I) - r_1 I\circ(\mathrm{D} - r_2 I)\\ &= \mathrm{D}\circ\mathrm{D} - \mathrm{D}\circ r_2 I - r_1 I\circ\mathrm{D} - r_1 I\circ(-r_2 I)\\ &= \mathrm{D}^2 - r_2\mathrm{D} - r_1\mathrm{D} - r_1 r_2 I = L\ .)\end{aligned}$$

Now, compute

$$(\mathrm{D} - r_1 I)\left(e^{r_1 x}\right) = \mathrm{D}\left(e^{r_1 x}\right) - r_1 e^{r_1 x} = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0.$$

Thus this factoring of L as

$$L = (\mathrm{D} - r_1 I)\circ(\mathrm{D} - r_2 I) = (\mathrm{D} - r_2 I)\circ(\mathrm{D} - r_1 I)$$

is (another reason) why $e^{r_1 x}, e^{r_2 x}$ both satisfy $L(y) = 0$ in case $r_1 \neq r_2$ are roots of the characteristic polynomial, since each exponential is transformed into the zero function by one of the linear factors of L. The case $r_1 = r_2$ is the double root case,

$$L = (D - r_1 I) \circ (D - r_1 I) = (D - r_1 I)^2 .$$

More generally, if the characteristic polynomial for the n^{th} order homogeneous DE factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

then the corresponding operator L factors as a composition

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m} .$$

w9.4)

a) Let $f(x)$ be any differentiable function. Check that

$$(D - r_1 I)\left(f(x) e^{r_1 x}\right) = f'(x) e^{r_1 x} .$$

b) Deduce that

$$(D - r_1 I) e^{r_1 x} = 0$$

$$(D - r_1 I)\left(x e^{r_1 x}\right) = e^{r_1 x}$$

$$(D - r_1 I) x^2 e^{r_1 x} = 2 x e^{r_1 x} .$$

$$(D - r_1 I) x^k e^{r_1 x} = k x^{k-1} e^{r_1 x} .$$

c) Use your work from b to explain why

$$(D - r_1 I)^2 \left(x e^{r_1 x}\right) = 0$$

$$(D - r_1 I)^3 x^2 e^{r_1 x} = 0 .$$

$$(D - r_1 I)^k x^{k-1} e^{r_1 x} = 0, \ k \in \mathbb{N} .$$

d) Explain why if L has a factor

$$(D - r_1 I)^{k_1}$$

with $k_1 > 1$, then the k_1 functions

$$e^{r_1 x}, x e^{r_1 x}, \ldots, x^{k_1 - 1} e^{r_1 x}$$

as we wrote down in the algorithm for finding a solution space basis, do all indeed satisfy $L(y) = 0$.