Math 2250-004

Week 8 concepts and homework, due Friday October 21.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

4.2: *subspaces: Carry over problems from last homework,* 3,**4**,5, **6**, **9**, 15, 17, 24, **27**, 29; add additional problems **11, 12** .

4.4: *bases for subspaces:* 1, **2**, 3, **4** ,**6**, 15.

w8.1) Consider the matrix $A_{3 \times 6}$ given by

$$A := \begin{bmatrix} 2 & 1 & -1 & 0 & 1 & 4 \\ 4 & 2 & 0 & 4 & -3 & -2 \\ 2 & 1 & 2 & 6 & 2 & 6 \end{bmatrix}.$$

The reduced row echelon of this matrix is

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

w8.1a) Find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^6 \ s.t.\ A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. What is the dimension of this subspace?

w8.1.b) Find a basis for the span of the six columns of A. Note that this a subspace of $\mathbb{R}^3$. Pick your basis so that it uses some (but not all!) of the columns of A.

w8.1c) The dimensions of the two subspaces in parts a,b add up to 6, the number of columns of A. This is an example of a general fact, known as the "rank plus nullity theorem". To see why this is always true, consider any matrix $B_{m \times n}$ which has m rows and n columns. As in parts a,b consider the homogeneous solution space

$$W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ s.t.\ B\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n$$

and the column space

$$V = span\{col_1(B), col_2(B), ...col_n(B)\} = \{B\,\underline{\mathbf{c}},\ s.t.\ \underline{\mathbf{c}} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Let the reduced row echelon form of B have k leading 1's, with $0 \le k \le m$. Explain what the dimensions of W and V are in terms of k and n, and then verify that

$$dim(W) + dim(V) = n$$

must hold.

Remark: The dimension of the column space V above is called the column rank of the matrix. The homogeneous solution space W is often called the nullspace of A, and its dimension is sometimes called the nullity. That nomenclature is why the theorem is called the "rank plus nullity theorem". You can read more about this theorem, which has a more general interpretation, at wikipedia (although the article gets pretty dense after the first few paragraphs).

w8.2) Recall that because of how matrix multiplication works, column dependencies for a matrix A, i.e.

$$c_1 col_1(A) + c_2 col_2(A) + ... + c_n col_n(A) = \underline{\mathbf{0}}$$

are equivalent to homogeneous solutions $\underline{c} = [c_1, c_2, ..., c_n]^T$ to the matrix equation

$$A\,\underline{c} = \underline{\mathbf{0}}.$$

As we discuss in class using the notes from Monday October 17, this implies that column dependencies in A correspond exactly to column dependencies in the reduced row echelon form of A. This fact, and the four conditions that a reduced row echelon form matrix must satisfy, means that one can deduce the reduced row echelon form of a matrix A by knowing the column dependencies of A. This is the reason each matrix has only one reduced row echelon form no matter the order of row operations that one uses to compute it. (We've been using this fact since the start of Chapter 3, but we never explained why it was true before.) As an example of this fact, let a matrix $A_{4 \times 6}$ have the following list of column dependencies:

$col_1(A) = \underline{\mathbf{0}}$ (i.e. column one is linearly dependent).

$col_2(A) \neq \underline{\mathbf{0}}$ (i.e. column two is linearly independent)

$col_3(A) = -3\, col_2(A)$

$col_4(A)$ is not a linear combination of $col_2(A)$ (i.e. not a scalar multiple)

$col_5(A) = -\, col_2(A) + 2\, col_4(A)$

$col_6(A)$ is not a linear combination of $col_2(A), col_4(A)$.

What must the reduced row echelon form of A be? Explain.

Chapter 5 problems are postponed until next week:

5.1: Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.

1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 1, y'(1) = 7$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

w8.3a) In 5.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where the constant r is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of r for which $y(x)$ will satisfy the DE. (See Theorem 5 in the text.)

w8.3b) In 5.1.10 above, the text tells you that $y_1(x) = e^{5x}, y_2(x) = x\,e^{5x}$ are two independent solutions to the second order homogeneous differential equation $y'' - 10y' + 25y = 0$. Follow the procedure in part a of trying for solutions of the form $y(x) = e^{rx}$, and then use the repeated roots Theorem 6 in the text, to recover these two solutions.

5.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.

1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26