solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

in-phase, equal amplitude mode

$x_1(t) = x_2(t)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\cos(\omega_1 t - \alpha_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\cos(\omega_2 t - \alpha_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

out of phase mode,

faster

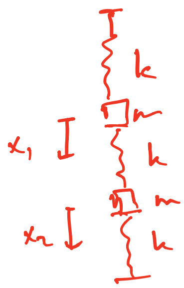
$w_2 = \sqrt{3}\,w_1$

$$= \left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos(\omega_2 t) + c_4\sin(\omega_2 t)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$x_1(t) =$

$x_2(t) =$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.



$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

find c_1, c_2, c_3, c_4 to solve each IVP

$$\begin{cases} x_1(0) = a_1, & x_1'(0) = a_2 \\ x_2(0) = b_1, & x_2'(0) = b_2 \end{cases}$$

$x_1(t) = c_1\cos w_1 t + c_2 \sin w_1 t + c_3 \cos w_2 t + c_4 \sin w_2 t$

$x_1(0) = a_1 = c_1 + c_3$

$x_1'(0) = a_2 = w_1 c_2 + w_2 c_4$

$x_2(0) = b_1 = c_1 - c_3$

$x_2'(0) = b_2 = w_1 c_2 - w_2 c_4$

solve for c_1 & c_3 from a_1 & b_1

solve for c_2 & c_4 from a_2, b_2

Experiment: Although we won't have time in class to measure the spring constants, I've measure them earlier. We can predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to Exercise 2, and then check our predictions like we did for the single mass-spring configuration, I have brought along a demonstration so that we can see these two vibrations.

> $Digits := 5:$

$k := \dfrac{.05 \cdot 9.806}{.153};$

$\omega 1 := \sqrt{\dfrac{k}{.05}};\ T1 := evalf\left(\dfrac{2 \cdot \pi}{\omega 1}\right);$

$\omega 2 := \sqrt{3.0} \cdot \omega 1;\ T2 := evalf\left(\dfrac{2 \cdot \pi}{\omega 2}\right);$

$\omega_1 = \sqrt{\frac{k}{m}}$ ← slow

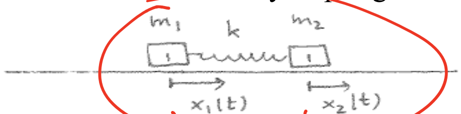
$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_1$ ← fast

$$k := 3.2046$$

$$\left.\begin{array}{l} \omega 1 := 8.0057 \\ T1 := 0.78483 \end{array}\right\} \text{slow}$$

$$\left.\begin{array}{l} \omega 2 := 13.867 \\ T2 := 0.45311 \end{array}\right\} \text{fast} \qquad (2)$$

Exercise 4) Consider a train with two cars connected by a spring:



equal length

See CO_2 vibration problem
also HW

4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}(x_2 - x_1)$$
$$x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

$\oplus$ stretch out

check:

$\cancel{m_1} x_1'' = \frac{k}{m_1}(x_2 - x_1)$.

$\cancel{m_2} x_2'' = -\frac{k}{m_2}(x_2 - x_1)$

$\ominus$

4b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$E_{\lambda=0}$ eigenvect. $\vec{x}'' = A\vec{x}$

> $Eigenvectors\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right)$;

$\omega_1 = 0$

$\omega_2 = \sqrt{-\lambda_2}$

$$\begin{bmatrix} \lambda_1 = 0 \\ \lambda_2 = -\frac{k(m_1 + m_2)}{m_2 m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \quad (3)$$

$\vec{v}$ $\quad \vec{w}$ λ_2 eigenvect

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos\omega_2 t + c_2 \sin\omega_2 t)\begin{bmatrix} -\frac{m_2}{m_1} \\ 1 \end{bmatrix} + (c_3 + c_4 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

out of phase vibration

$m_1 = m_2$, amplitudes $=$.

$m_1 > m_2$ x_1 amplitude is smaller than x_2 amplitude.

train moving @ ~~speed~~ vel. c_4, started @ c_3

$\vec{x}'' = A\vec{x}$

$\lambda_2 = 0$, $A\vec{w} = \vec{0}$

soln $\vec{x} = f(t)\vec{w}$

plug in:

$\vec{x}''(t) = \boxed{f''(t)\vec{w}}$

$A\vec{x} = A(f(t)\vec{w})$

$= f(t)A\vec{w}$

$= f(t)\vec{0} = \boxed{\vec{0}}$

$\Rightarrow f(t) = c_3 + c_4 t$

(need $f''(t) = 0$)

Math 2250-004
Tues Dec 6
7.4 Forced mass-spring systems

Warm up exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

1st order syst

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

2nd order sys.

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Announcements:

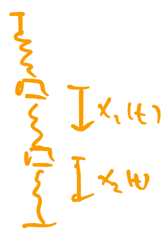
- Lab: tech. fine for eigendata
- Lab this week is for Friday HW (can turn HW in there)
- Review for exam tomorrow in class, also go over old exam next week Tues 4:30-6:00

eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\lambda = -5$, $\underline{v} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\underline{v} = [2, 1]^T$ is an eigenvector

$E_{\lambda=-5} = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ $\qquad$ $E_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} ? \\ 1 \end{bmatrix}\right\}$



1a $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(e^{\lambda t} \vec{v})$

1b $\omega = \sqrt{-\lambda}$

$\cos\omega t\, \vec{v}, \sin\omega t\, \vec{v}$

out of phase

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \boxed{\left(c_1 \cos(\sqrt{5}\, t) + c_2 \sin\sqrt{5}\, t\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

$\lambda = -5$
$\omega = \sqrt{5}$

in-phase

$\boxed{+\left(c_3 \cos t + c_4 \sin t\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

$\lambda = -1$
$\omega = 1$

$x_1(t) = (c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t)(-2) + (c_3 \cdots)2$
$x_2(t) = (\quad '' \quad)(1) + (- - -)(1)$

Forced oscillations (still undamped):

$$M^{-1}\left(M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{F}}(t) \right)$$

$$\Rightarrow \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\ .$$

$A = M^{-1}A,$

If the forcing is sinusoidal,

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\,\underline{\boldsymbol{G}}_0$$

$$\Rightarrow \boxed{\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\,\underline{\boldsymbol{F}}_0}$$

with $\underline{\boldsymbol{F}}_0 = M^{-1}\underline{\boldsymbol{G}}_0$.

$\vec{x}'' - A\vec{x} = (\cos\omega t)\vec{F}_0$ non-homog.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)\ ,$$

and we've been discussing how to find the homogeneous solutions $\underline{\boldsymbol{x}}_H(t)$. ✓

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega\, t)\ \underline{\boldsymbol{c}}$$

where the vector $\underline{\boldsymbol{c}}$ is what we need to find.

RHS = $\cos\omega t\ \vec{F}_0$
even derivs.
$\vec{x}_p(t) = \cos\omega t\ \vec{d}$
should work except when ω = one of natural frequencies

Exercise 1) Substitute the guess $\underline{\boldsymbol{x}}_P(t) = \cos(\omega\, t)\ \underline{\boldsymbol{c}}$ into the DE system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\,\underline{\boldsymbol{F}}_0$$

to find a matrix algebra formula for $\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

- $\vec{x}'' = A\vec{x} + \cos\omega t\ \vec{F}_0$

$\vec{x}_p = (\cos\omega t)\vec{d}$

LHS : $-\omega^2(\cos\omega t)\,\vec{d}$

RHS $\quad A(\cos\omega t)\vec{d} + \cos\omega t\ \vec{F}_0$

$\div \cos\omega t$:

LHS = RHS : $\quad -\omega^2\vec{d} = A\vec{d} + \vec{F}_0$

$-\vec{F}_0 = A\vec{d} + \omega^2\vec{d}$

$-\vec{F}_0 = (A + \omega^2 I)\vec{d}$

$(A + \omega^2 I)^{-1}(-\vec{F}_0) = \vec{d}$

eigenvector eqn
$(A - \lambda I)\vec{v} = \vec{0}$
as long as $\lambda \neq \lambda_i$
$(A - \lambda I)$ has an inverse

Solution:

$$\boxed{\underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}}_0\ .}$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.