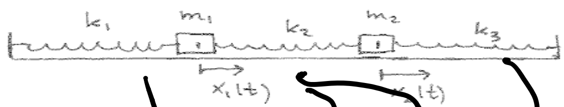
Math 2250-004
Week 15: Section 7.4, mass-spring systems.
These are notes for Monday and Tuesday (and possibly for part of Wednesday). There will also be course review notes for Wednesday, posted later.

Mon Dec 5
7.4 Mass-spring systems and untethered mass-spring trains.
In your homework and lab for this week you study special cases of the spring systems below, with no damping. Although we draw the pictures horizontally, they would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.




HW. Stretch amts: spring 1: $x_1$; 2: $x_2 - x_1$; 3: $-x_2$

Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Newton: $m_1 x_1'' = (-k_1 - k_2)x_1 + k_2 x_2$

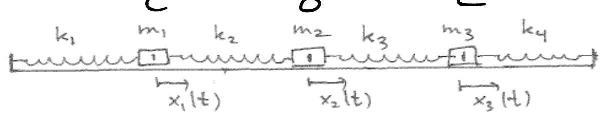
• initial pos. • initial vel

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

4 : 4 free parameters (IC's).
or equivalent to IVP for $1^{st}$ order linear homogeneous system for 4 fns ($x_1, x_1', x_2, x_2'$): $\vec{x}' = A\vec{x}$.

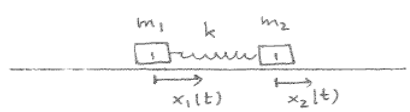
1b) What if one had a configuration of $n$ masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:

dim = 6




$CO_2$ model in lab. ($k_1 = k_4 = 0$)

dim = 4



We can write the system of DEs for the system at the top of page 1 in matrix-vector form: entries

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$m_1 x_1'' = (-k_1 - k_2)x_1 + k_2 x_2$

$m_2 x_2'' = k_2 x_1 \; (-k_1 - k_3) x_2$

We denote the diagonal matrix on the left as the "mass matrix" $M$, and the matrix on the right as the spring constant matrix $K$ (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as Chptr 5). ($mx'' = -kx$)

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}}\,.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$\div m_1: \; x_1'' = \left(-\dfrac{(k_1+k_2)}{m_1}\right)x_1 + \dfrac{k_2}{m_1}x_2$

$\div m_2: \; x_2'' = \dfrac{k_2}{m_2}x_1 - \dfrac{(k_1+k_3)}{m_2}x_2$

which we write as

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\,.$$

(You can think of $A$ as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix $M$. In all cases:

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\,, \text{ with } A = M^{-1}K\,.$$

$$M^{-1}(M\vec{x}'') = M^{-1}(K\vec{x})$$

$$\vec{x}'' = A\vec{x}$$

$$(A = M^{-1}K)$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

could convert to 1st order system.

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\text{try}\quad \underline{\boldsymbol{x}}(t) = e^{r\,t}\underline{\boldsymbol{v}}$$

We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function. But let's try the substitution directly, in analogy to what we did for higher order single linear differential equations back in Chapter 5.

Now, in the present case of systems of masses and springs we are assuming there is no damping. Thus, the total energy - consisting of the sum of kinetic and potential energy - will always be conserved. Any two complex solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{r\,t}\boldsymbol{v}^{\pm} = e^{(a\,\pm\,\omega i)t}\boldsymbol{v}^{\pm}$$

$r = (a \pm \omega i)$, $\vec{v}^{\pm} = \vec{u} \pm i\vec{w}$

would yield two real solutions $\underline{\boldsymbol{X}}(t), \underline{\boldsymbol{Y}}(t)$ where

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{X}}(t) \pm i\,\underline{\boldsymbol{Y}}(t)\ .$$

what abt r? $a > 0$

Because of conservation of energy ($TE = KE + PE$ must be constant), neither $\underline{\boldsymbol{X}}(t)$ nor $\underline{\boldsymbol{Y}}(t)$ can grow or decay exponentially - if a solution grew exponentially the total energy would also grow exponentially; if it decayed exponentially the total energy would decay exponentially. SO, we must have $a = 0$. In other words, in order for the total energy to remain constant we must actually have

$$\underline{\boldsymbol{x}}(t) = e^{\pm i\,\omega\,t}\underline{\boldsymbol{v}}^{\pm}.$$

$r = i\omega$, $\vec{x}(t) = e^{i\omega t}\vec{v}$

Substituting this $x(t)$ into the homogeneous DE

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

yields the necessary condition

$\vec{x}'(t) = i\omega e^{i\omega t}\vec{v}$

RHS $\vec{x}''(t) = -\omega^2 e^{i\omega t}\vec{v}$

LHS $Ae^{i\omega t}\vec{v} = e^{i\omega t}A\vec{v}$

$$-\omega^2 e^{i\,\omega\,t}\underline{\boldsymbol{v}} = e^{i\,\omega\,t}A\,\underline{\boldsymbol{v}}\quad .$$

So $\underline{\boldsymbol{v}}$ must be an eigenvector, with non-positive eigenvalue $\lambda = -\omega^2$,

$$A\,\underline{\boldsymbol{v}} = -\omega^2\,\underline{\boldsymbol{v}}\,.$$

And since row reduction will find real eigenvectors for real eigenvalues, we can find eigenvectors $\underline{\boldsymbol{v}}$ with real entries. And the two complex solutions

$\lambda = -\omega^2 \in \mathbb{R}$ ⟹ real eigenvectors $\vec{v}$

$$\underline{\boldsymbol{x}}(t) = e^{\pm\,i\omega t}\underline{\boldsymbol{v}} = \cos(\omega\,t)\underline{\boldsymbol{v}} \pm i\sin(\omega\,t)\underline{\boldsymbol{v}}$$

↑ real ↑ real ↑ real

yield the two real solutions

$$\underline{\boldsymbol{X}}(t) = \cos(\omega\,t)\underline{\boldsymbol{v}},\qquad \underline{\boldsymbol{Y}}(t) = \sin(\omega\,t)\underline{\boldsymbol{v}}\,.$$

So, we skip the exponential solutions altogether, and go directly to finding homogeneous solutions of the form above. We just have to be careful to remember that $\underline{\boldsymbol{v}}$ is an eigenvector with eigenvalue $\lambda = -\omega^2$, i.e. $\omega = \sqrt{-\lambda}$.

shortcut

OR directly: try $\vec{x} = \cos\omega t\,\vec{v}$

$\vec{x}'' = -\omega^2\cos(\omega t)\vec{v}$

$A\vec{x} = \cos(\omega t)A\vec{v}$ if $A\vec{v} = -\omega^2\vec{v}$

(same for $\vec{x} = (\sin\omega t)\vec{v}$)

Note: In analaogy with the scalar undamped oscillator DE

$$x''(t) + \omega_0^2\, x = 0$$

where we could read off and check the solutions

$$\cos(\omega_0 t),\ \sin(\omega_0 t)$$

knew right away

directly without going throught the characteristic polynomial, it is easy to check that

$$\cos(\omega\, t)\,\underline{\boldsymbol{v}},\ \sin(\omega\, t)\,\underline{\boldsymbol{v}}$$

each solve the conserved energy mass spring system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

as long as

$$-\omega^2\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}.$$

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

If $A_{n\times n}$ is a diagonalizable matrix and if all of its eigenvalues are negative, then for each eigenpair $(\lambda_j, \underline{\boldsymbol{v}}_j)$ there are two linearly independent solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos(\omega_j\, t)\,\underline{\boldsymbol{v}}_j \qquad \boldsymbol{y}_j(t) = \sin(\omega_j\, t)\,\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

(1) Find eigenvalues & eigenvectors for A [same as for 1st order sys]

(2) solns are $\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$

$\omega^2 = -\lambda$

$\omega = \sqrt{-\lambda}$

different from 1st order, solns are NOT $e^{\lambda t}\vec{v}$ !!

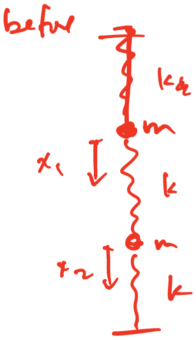
Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $(c_1 + c_2\, t)\,\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

system from before




a) Find the eigendata for the matrix

$$B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix $A$ which is $\frac{k}{m}$ times this matrix.

c) Find the 4 - dimensional solution space to this two-mass, three-spring system.

a) $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = (\lambda+3)(\lambda+1)$ roots $\lambda = -1, -3$

$\lambda = -1$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$, $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b) $A = \frac{k}{m} B$

$A\vec{v} = \frac{k}{m} B\vec{v} = \frac{k}{m}(-1\vec{v})$

$A\vec{v} = -\frac{k}{m}\vec{v}$. $\vec{v}$ eigenvect of $A$, $\lambda = -1 \cdot \frac{k}{m}$

$B\vec{v} = \lambda\vec{v}$

$A = cB$ then $A\vec{v} = cB\vec{v} = (c\lambda)\vec{v}$

- mult matrix by const. same eigenvectors, but eigenvalues are multiplied by same constant

for $\lambda = -1 \cdot \frac{k}{m} = -\frac{k}{m}$ $\qquad$ $\lambda = -3 \cdot \frac{k}{m} = -\frac{3k}{m}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\omega_1 = \sqrt{-\lambda} = \sqrt{\frac{k}{m}}$ $\qquad$ $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\omega_2 = \sqrt{-\lambda} = \sqrt{\frac{3k}{m}} = \sqrt{3}\,\omega_1$

c) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos\omega_2 t + c_4 \sin\omega_2 t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3k}{m}}$. The general solution can be written as

in-phase, equal amplitude mode: $x_1(t) = x_2(t)$

out of phase mode, faster: $\omega_2 = \sqrt{3}\,\omega_1$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\cos(\omega_1 t - \alpha_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\cos(\omega_2 t - \alpha_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1\cos(\omega_1 t) + c_2\sin(\omega_1 t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos(\omega_2 t) + c_4\sin(\omega_2 t)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

find $c_1, c_2, c_3, c_4$ to solve each IVP

Experiment: Although we won't have time in class to measure the spring constants, I've measure them earlier. We can predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to Exercise 2, and then check our predictions like we did for the single mass-spring configuration, I have brought along a demonstration so that we can see these two vibrations.

> $Digits := 5:$
> $k := \frac{.05 \cdot 9.806}{.153};$
> $\omega 1 := \sqrt{\frac{k}{.05}};\ T1 := evalf\left(\frac{2 \cdot \pi}{\omega 1}\right);$
> $\omega 2 := \sqrt{3.0} \cdot \omega 1;\ T2 := evalf\left(\frac{2 \cdot \pi}{\omega 2}\right);$

$$k := 3.2046$$

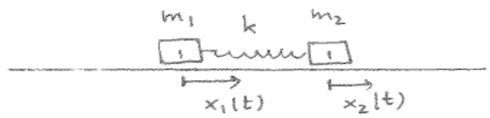
$$\omega 1 := 8.0057$$

$$T1 := 0.78483$$

$$\omega 2 := 13.867$$

$$T2 := 0.45311 \quad \textbf{(2)}$$

Exercise 4) Consider a train with two cars connected by a spring:




4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}\left(x_2 - x_1\right)$$

$$x_2'' = -\frac{k}{m_2}\left(x_2 - x_1\right)$$

4b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

> $Eigenvectors\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right);$

$$\begin{bmatrix} 0 \\ -\frac{k\,(m_1+m_2)}{m_2\,m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \qquad \textbf{(3)}$$