Math 2250-004
Wednesday, December 7
Course review

I fixed the date in our on-line copy

Final exam: Friday December 15, 8:00 a.m. -10:00 a.m. (I will let you start working at 7:45 and stay until 10:15 if you wish.) This is the official University time and location – our MWF lecture room JWB 335 (here). As usual the exam is closed book and closed note, and the only sort of calculator that is allowed is a simple scientific one. Your cell phones need to be put away for the entire exam. You will be provided the Laplace Transform table from the front cover of our text. The algebra and math on the exam should all be doable by hand.

Review of previous final exam practice exams: Next week Tuesday December 13 4:00-6:00 p.m. Location TBA.

something e.g., 10–12 Thursday

I'll let you know when these are finalized.

The final exam will be comprehensive, but weighted to more recent material. Rough percentage ranges per chapter are below – these percentage ranges add up to more than 100% because many topics and problems span several chapters. Also consult the "course learning objectives" from our syllabus (next page). Note that we did not cover Chapter 9.

Chapters

- 1-2: 10-20% first order DEs
- 3-4: 20-40% matrix algebra and vector spaces
- 5: 15-30% linear differential equations and applications
- 6.1-6.2: 15-30% eigenvalues and eigenvectors, including complex case
- 7.1-7.4: 20-40% linear systems of differential equations and applications
- 10.4-10.5, EP 7.6: 20-40% Laplace transform

Should say 10.1–10.5 !!

On the third page is a more detailed list of the topics we've investigated this semester. They are more inter-related than you may have realized at the time, so we'll discuss the connections in class. Then we'll look at an extended problem that highlights these connections and connects perhaps 70% of the key ideas in this course. We won't have time to work out everything in class, but filling in the details might help you consolidate these ideas.

Learning Objectives for 2250

The goal of Math 2250 is to master the basic tools and problem solving techniques important in differential equations and linear algebra. These basic tools and problem solving skills are described below.

The essential topics

Be able to model dynamical systems that arise in science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental (input-output) analysis, Newton's laws, conservation of energy and ~~Kirchoff's law~~.

Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering. Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and ~~numerical~~ approximations to initial value problems via slope fields.

slope fields possible but numerical methods

3 Become fluent in matrix algebra techniques, in order to be able to compute the solution space to linear systems and understand its structure; by hand for small problems and with technology for large problems.

4, 5, 7 Be able to use the basic concepts of linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear equations, linear differential equations, and linear systems of differential equations.

Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.

Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand the solutions to the basic unforced and forced mechanical and ~~electrical~~ oscillation problems.

Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, ~~electrical circuits~~, and related problems. Laplace for systems too

Be able to find eigenvalues and eigenvectors for square matrices. Apply these matrix algebra concepts to find the general solution space to first and second order constant coefficient homogeneous (Chapter 7) linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.

Chapter 7 ties together most of the previous course material that's what makes it so interesting, but also harder.

~~Chpt 9~~ Understand and be able to use linearization as a technique to understand the behavior of nonlinear autonomous dynamical systems near equilibrium solutions. Apply these techniques to non-linear mechanical oscillation problems and other systems of two first order differential equations, including interacting populations. Relate the phase portraits of non-linear systems near equilibria to the linearized data, in particular to understand stability.

Develop your ability to communicate modeling and mathematical explanations and solutions, using technology and software such as Maple, Matlab or internet-based tools as appropriate.

Problem solving fluency

Students will be able to read and understand problem descriptions, then be able to formulate equations modeling the problem usually by applying geometric or physical principles. Solving a problem often requires specific solution methods listed above. Students will be able to select the appropriate operations, execute them accurately, and interpret the results using numerical and graphical computational aids.

Students will also gain experience with problem solving in groups. Students should be able to effectively transform problem objectives into appropriate problem solving methods through collaborative discussion. Students will also learn how to articulate questions effectively with both the instructor and TA, and be able to to effectively convey how problem solutions meet the problem objectives.

1-2: first order DEs

- slope fields, Euler approximation
- phase diagrams for autonomous DEs
  - equilibrium solutions
  - stability
- existence-uniqueness thm for IVPs
- methods:
  - separable
  - linear
- applications
  - populations
  - velocity-acceleration models
  - input-output models

3-4 matrix algebra and vector spaces

- linear systems and matrices
- reduced row echelon form
- matrix and vector algebra
  - manipulating and solving matrix-vector equations for unknown vectors or matrices.
- matrix inverses
- determinants
- vector space concepts
  - vector spaces and subspaces
  - linear combinations
  - linear dependence/independence
  - span
  - basis and dimension
  - linear transformations
    - aka superposition
    - fundamental theorem for solution space to L(y)=f when L is linear

5 Linear differential equations

- IVP existence and uniqueness
- Linear DEs
  - Homogeneous solution space, its dimension, and why
  - superposition, $\underline{x}(t)=\underline{x}_P+\underline{x}_H$
- Constant coefficient linear DEs
  - $\underline{x}_H$ via characteristic polynomial
    - Euler's formula, complex roots
  - $\underline{x}_P$ via undetermined coefficients
  - solving IVPs
- applications:
  - mechanical configurations
    - unforced: undamped and damped
      - cos and sin addition angle formulas and amplitude-phase form
    - forced undamped: beating, resonance
    - forced damped: $\underline{\mathbf{x}}_{sp}+\underline{\mathbf{x}}_{tr}$, practical resonance
    - ~~RLC circuits~~
    - Using conservation of total energy (=KE+PE) to derive equations of motion, especially for mass-spring and pendulum

6.1-6.1 eigenvalues, eigenvectors (eigenspaces), diagonalizable matrices; real+complex eigendata.

7.1-7.4 linear systems of DEs

- first order systems of DEs and tangent vector fields.
- existence-uniqueness thm for first order IVPs
- superposition, $\underline{\mathbf{x}}=\underline{\mathbf{x}}_P+\underline{\mathbf{x}}_H$
- dimension of solution space for $\underline{\mathbf{x}}_H$.
- equivalence of DE IVPs or systems to first order system IVPs.
- Constant coefficient systems and methods for $\underline{\mathbf{x}}=\underline{\mathbf{x}}_P+\underline{\mathbf{x}}_H$:
  - $\underline{\mathbf{x}}'(t)=A\underline{\mathbf{x}}$
  - $\underline{\mathbf{x}}'(t)=A\underline{\mathbf{x}}+\underline{\mathbf{f}}(t)$
  - $\underline{\mathbf{x}}''(t)=A\underline{\mathbf{x}}$ (from conservative systems)
  - $\underline{\mathbf{x}}''(t)=A\underline{\mathbf{x}}+\underline{\mathbf{f}}(t)$
- applications: phase portrait interpretation of unforced oscillation problems; input-output modeling; forced and unforced mass-spring systems and phenomena.

10.1-10.5, EP7.6: Laplace transform

- definition, for direct computation
- using table for Laplace and inverse Laplace transforms … including for topics before/after the second midterm, i.e. on/off and impulse, forcing, convolution solutions
- Solving linear DE (or system of DE) IVPs with Laplace transform.

orange : $\exists!$
red : applications
black : core vector space, linear algebra
purple : $L(\vec{x})=f$ L differential op.
blue : "Linear"   fundamental theorem for solving $L(y)=f$, namely $y=y_P+y_H$. Also, superposition

We can illustrate many ideas in this course, and how they are tied together by studying the following two differential equations in as many ways as we can think of.

x''(t) + 5 x'(t) + 4 x(t) = 0

- basis of soln space made out of exponentials or exp-trig or trig, via $p(r)=0$
- physical phenomenon (overdamped)
- solve via Laplace transform. then your 2 free parameters are x_0 & v_0.
- IVP
- as first order system for $\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$:

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} v \\ -4x-5v \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

- solve sys via eigenvalue-eigenvector analysis --- $p(\lambda) = |A - \lambda I|$ is multiple of $p(\lambda)$ from Chapter 5
  first component $x(t)$ of solution to 1st order system solves 2nd order DE
- phase portrait for $\begin{bmatrix} x \\ v \end{bmatrix}$ shows parametric curve points $\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$

x''(t) + 5 x'(t) + 4 x(t) = 3 cos(2t)

- $x = x_p + x_H$
  undetermined coeff's for x_p.
- IVP
- Laplace transform (and can invert to recover $x(t)$ by partial fractions or convolutions, for the part of soln satisfying $x(0)=0$, $x'(0)=0$
- as 1st order system

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \cos 2t \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$\vec{x} = \vec{x}_p + \vec{x}_H$

$\vec{x}_p$ by undetermined coefficients

$\vec{x}_p = (\cos 2t)\,\vec{a} + (\sin 2t)\,\vec{b}$.