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Math 2250-4
Super Quiz 3
December 6, 2013
SOLUTIONS

*1) Solve **one** of the two following Laplace transform problems. There is a Laplace transform table at the end of the superquiz. If you try both problems, indicate clearly which one you wish to have graded.*

(6 points)

1a) Find an integral convolution formula for the solution to this forced oscillation initial value problem. You do not need to evaluate the convolution integral:

$$x''(t) + 2x'(t) + 5x(t) = e^{-3t}$$
$$x(0) = 0$$
$$x'(0) = 0$$

Solution: Take the Laplace transform of the IVP, for the solution $x(t)$:

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{1}{s+3}$$
$$X(s)\left(s^2 + 2s + 5\right) = \frac{1}{s+3}$$
$$X(s) = \frac{1}{(s+1)^2 + 4}\,\frac{1}{s+3} = Z(s)F(s)$$

for $f(t) = \mathrm{e}^{-3t}$ and transfer funtion

$$z(t) = \frac{1}{2}\mathrm{e}^{-t}\sin(2t).$$

Thus by the convolution theorem,

$$x(t) = z * f(t) = \int_0^t z(\tau)f(t-\tau)\,\mathrm{d}\tau = \int_0^t \frac{1}{2}\mathrm{e}^{-\tau}\sin(2\tau)\mathrm{e}^{-3(t-\tau)}\,\mathrm{d}\tau.$$

(You could also write the convolution integral in reverse order.)

For fun, here's a technology check:

> $\int_0^t \frac{1}{2}\cdot\mathrm{e}^{-\tau}\cdot\sin(2\cdot\tau)\cdot\mathrm{e}^{-3\cdot(t-\tau)}\,\mathrm{d}\tau;$ *#actual solution*

$$\frac{1}{8}\left(1 + 2\,\mathrm{e}^{2t}\sin(t)\cos(t) - 2\,\mathrm{e}^{2t}\cos(t)^2 + \mathrm{e}^{2t}\right)\mathrm{e}^{-3t} \qquad \textbf{(1)}$$

>

The symbolic solution above shows how technology doesn't always simplify effectively. We may rewrite that convolution output as

$$x(t) = \frac{1}{8}\mathrm{e}^{-3t} + \frac{1}{8}\mathrm{e}^{-t}\left(2\sin(t)\cos(t) - 2\cos(t)^2 + 1\right)$$

$$= \frac{1}{8}e^{-3t} + \frac{1}{8}e^{-t}(\sin(2t) - \cos(2t))$$

which is how we would have found it using Chapter 5, via $x = x_P + x_H$, or if we'd done partial fractions on the Laplace transform expression

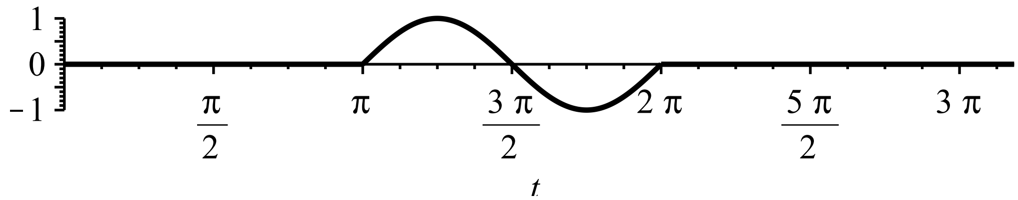
$$X(s) + \frac{1}{(s^2 + 2s + 5)(s + 1)} = \frac{As + B}{s^2 + 2s + 5} + \frac{C}{s + 1}$$

instead of using the convolution integral.

<u>1b</u>) *Solve the initial value problem below for an undamped mass-spring configuration subject to impulse forces at time* $t = \pi$ *and* $t = 2\pi$.

$$x''(t) + 4x(t) = 2 \cdot \delta(t - \pi) - 2 \cdot \delta(t - 2\pi)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

Hint: The solution has this graph:



<u>Solution</u>: Take the Laplace transform of the IVP, for the solution $x(t)$:

$$s^2X(s) + 4X(s) = 2e^{-\pi s} - 2e^{-2\pi s}$$
$$\Rightarrow X(s) = \frac{2e^{-\pi s}}{s^2 + 4} - \frac{2e^{-2\pi s}}{s^2 + 4}.$$

Since the inverse Laplace transform of $e^{-as}F(s)$ is $u(t - a)f(t - a)$ and since $F(s) = \frac{2}{s^2 + 4}$ has inverse Laplace transform $f(t) = \sin(2t)$ we deduce that

$$x(t) = 2u(t - \pi)\sin(2(t - \pi)) - 2u(t - 2\pi)\sin(2(t - 2\pi)).$$

Since $\sin(2t)$ has period π we may simplify:

$$x(t) = 2u(t - \pi)\sin(2t) - 2u(t - 2\pi)\sin(2t).$$

This can also be written using "piecewise" notation:

$$x(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin(2t), & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

which is what the graph shows.

2) Find the eigenvalues and eigenvectors (eigenspace bases) for the following matrix

$$\begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix}$$

Hint: if you compute your characteristic polynomial correctly you will find that the eigenvalues are negative integers.

(5 points)

Solution: The characteristic polynomial is given by

$$p(\lambda) = \begin{vmatrix} -2-\lambda & 5 \\ 1 & -6-\lambda \end{vmatrix} = (\lambda+2)(\lambda+6) - 5 = \lambda^2 + 8\lambda + 7 = (\lambda+1)(\lambda+7).$$

Thus the eigenvalues (roots of the characteristic polynomial) are $\lambda = -1, -7$.

Eigenspace for $\lambda = -1$: We wish to find a basis for the solution space to $(A + I)\underline{\mathbf{v}} = \underline{\mathbf{0}}$:

$$\left[\begin{array}{cc|c} -1 & 5 & 0 \\ 1 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

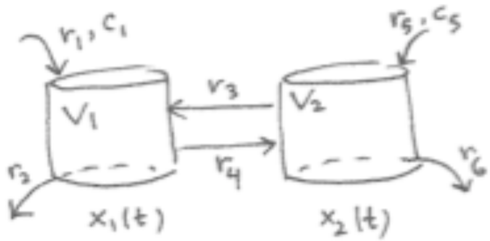
so $\underline{\mathbf{v}} = [5, 1]^T$ is an eigenvector (eigenspace basis).

Eigenspace for $\lambda = -7$: We wish to find a basis for the solution space to $(A + 7I)\underline{\mathbf{v}} = \underline{\mathbf{0}}$:

$$\left[\begin{array}{cc|c} 5 & 5 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $\underline{\mathbf{v}} = [1, -1]^T$ is an eigenvector (eigenspace basis).

3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t)$, $x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



3a) Suppose $r_1 = r_2 = r_3 = r_4 = 100,\ r_5 = r_6 = 20\ \frac{m^3}{hour}$. *Explain why the volumes* $V_1(t), V_2(t)$ *remain constant.*

(2 points)

Solution: The volumes will be constant if and only if their time derivatives are identically zero:

$$V_1'(t) = r_1 + r_3 - r_2 - r_4 = 100 + 100 - 100 - 100 = 0$$
$$V_2'(t) = r_4 + r_5 - r_3 - r_6 = 100 + 20 - 100 - 20 = 0.$$

3b) Using the flow rates above, incoming concentrations $c_1 = 0,\ c_5 = 1.4\ \frac{kg}{m^3}$, *volumes* $V_1 = 100\ m^3,\ V_2 = 20\ m^3$, *show that the amounts of solute* $x_1(t)$ *in tank 1 and* $x_2(t)$ *in tank 2 satisfy*

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 28 \end{bmatrix}.$$

(4 points)

Solution:

$$x_1'(t) = r_1 c_i + r_3 c_i - r_2 c_o - r_4 c_o = 100 \cdot 0 + 100\frac{x_2}{20} - 100\frac{x_1}{100} - 100\frac{x_1}{100} = -2x_1 + 5x_2$$

$$x_2'(t) = r_4 c_i + r_5 c_i - r_3 c_o - r_6 c_o = 100\frac{x_1}{100} + 20 \cdot 1.4 - 100\frac{x_2}{20} - 20\frac{x_2}{20} = x_1 - 6x_2 + 28.$$

3c) Verify that a particular solution to this system of differential equations is given by the constant vector function

$$\underline{\boldsymbol{x}}_P = \begin{bmatrix} 20 \\ 8 \end{bmatrix}.$$

(2 points)

Solution: For the constant solution $\underline{\boldsymbol{x}}_P = \begin{bmatrix} 20 \\ 8 \end{bmatrix}$ we have $x_P'(t) = \underline{\mathbf{0}}$. On the other hand, the right side of the differential equation system in this case is also $\underline{\mathbf{0}}$:

$$\begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 28 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} 20 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 28 \end{bmatrix} = \begin{bmatrix} -40+40 \\ 20-48 \end{bmatrix} + \begin{bmatrix} 0 \\ 28 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus this constant function makes the differential equation system true, i.e. it is a particular solution.

3d) Find the general solution to the system of differential equations in 3b. Hints: Use the particular solution from 3c as part of your solution, and notice that the matrix in this system is the same as the one in problem 2, so you can use the eigendata you found in that problem

(6 points)

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$$

Since the matrix in this problem is diagonalizable, be get a basis for the homogeneous solution space made out of functions of the form $e^{\lambda t}\underline{\mathbf{v}}$, using the eigendata from problem 2:

$$x_H(t) = c_1 e^{-t}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-7t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So, using the particular solution from part 3c,

$$\underline{\mathbf{x}}(t) = \begin{bmatrix} 20 \\ 8 \end{bmatrix} + c_1 e^{-t}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-7t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$