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# Math 2250-4
# Super Quiz 3
# December 6, 2013

1) Solve **one** of the two following Laplace transform problems. There is a Laplace transform table at the end of the superquiz. If you try both problems, indicate clearly which one you wish to have graded.

(6 points)

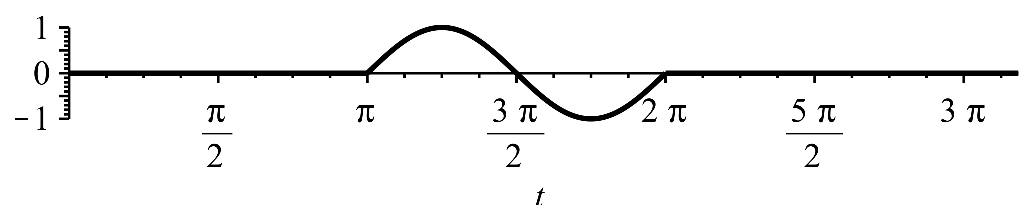
1a) Find an integral convolution formula for the solution to this forced oscillation initial value problem. You do not need to evaluate the convolution integral:

$$x''(t) + 2x'(t) + 5x(t) = e^{-3t}$$
$$x(0) = 0$$
$$x'(0) = 0$$

1b) Solve the initial value problem below for an undamped mass-spring configuration subject to impulse forces at time $t = \pi$ and $t = 2\pi$ .

$$x''(t) + 4x(t) = 2 \cdot \delta(t - \pi) - 2 \cdot \delta(t - 2\pi)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

Hint: The solution has this graph:



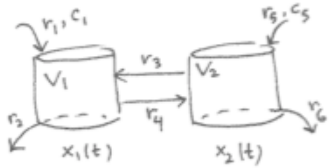
2) Find the eigenvalues and eigenvectors (eigenspace bases) for the following matrix

$$\begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix}$$

Hint: if you compute your characteristic polynomial correctly you will find that the eigenvalues are negative integers.

(5 points)

3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes $V_1$, $V_2$ and solute amounts $x_1(t)$, $x_2(t)$ respectively. The flow rates (volume per time) are indicated by $r_i$, $i = 1..6$. The two input concentrations (solute amount per volume) are $c_1$, $c_5$.




3a) Suppose $r_1 = r_2 = r_3 = r_4 = 100,\ r_5 = r_6 = 20\ \frac{m^3}{hour}$. Explain why the volumes $V_1(t)$, $V_2(t)$ remain constant.

(2 points)

3b) Using the flow rates above, incoming concentrations $c_1 = 0,\ c_5 = 1.4\ \frac{kg}{m^3}$, volumes $V_1 = 100\ m^3,\ V_2 = 20\ m^3$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 28 \end{bmatrix}.$$

(4 points)

3c) Verify that a particular solution to this system of differential equations is given by the constant vector function

$$\underline{\boldsymbol{x}}_P = \begin{bmatrix} 20 \\ 8 \end{bmatrix}.$$

(2 points)

3d) Find the general solution to the system of differential equations in 3b. Hints: Use the particular solution from 3c as part of your solution, and notice that the matrix in this system is the same as the one in problem 2, so you can use the eigendata you found in that problem

(6 points)

# Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

| Function | Transform | Function | Transform |
|---|---|---|---|
| $f(t)$ | $F(s)$ | $e^{at}$ | $\dfrac{1}{s-a}$ |
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ | $t^n e^{at}$ | $\dfrac{n!}{(s-a)^{n+1}}$ |
| $f'(t)$ | $sF(s)-f(0)$ | $\cos kt$ | $\dfrac{s}{s^2+k^2}$ |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ | $\sin kt$ | $\dfrac{k}{s^2+k^2}$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$ | $\cosh kt$ | $\dfrac{s}{s^2-k^2}$ |
| $\int_0^t f(\tau)\,d\tau$ | $\dfrac{F(s)}{s}$ | $\sinh kt$ | $\dfrac{k}{s^2-k^2}$ |
| $e^{at}f(t)$ | $F(s-a)$ | $e^{at}\cos kt$ | $\dfrac{s-a}{(s-a)^2+k^2}$ |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ | $e^{at}\sin kt$ | $\dfrac{k}{(s-a)^2+k^2}$ |
| $\int_0^t f(\tau)g(t-\tau)\,d\tau$ | $F(s)G(s)$ | $\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$ | $\dfrac{1}{(s^2+k^2)^2}$ |
| $tf(t)$ | $-F'(s)$ | $\dfrac{t}{2k}\sin kt$ | $\dfrac{s}{(s^2+k^2)^2}$ |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ | $\dfrac{1}{2k}(\sin kt + kt\cos kt)$ | $\dfrac{s^2}{(s^2+k^2)^2}$ |
| $\dfrac{f(t)}{t}$ | $\int_s^\infty F(\sigma)\,d\sigma$ | $u(t-a)$ | $\dfrac{e^{-as}}{s}$ |
| $f(t)$, period $p$ | $\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$ | $\delta(t-a)$ | $e^{-as}$ |
| $1$ | $\dfrac{1}{s}$ | $(-1)^{[\![t/a]\!]}$ (square wave) | $\dfrac{1}{s}\tanh\dfrac{as}{2}$ |
| $t$ | $\dfrac{1}{s^2}$ | $\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase) | $\dfrac{e^{-as}}{s(1-e^{-as})}$ |
| $t^n$ | $\dfrac{n!}{s^{n+1}}$ | | |
| $\dfrac{1}{\sqrt{\pi t}}$ | $\dfrac{1}{\sqrt{s}}$ | | |
| $t^a$ | $\dfrac{\Gamma(a+1)}{s^{a+1}}$ | | |