Math 2250 Week 4 Super Quiz

Name, UID, and section TA: ______________________________

Write your answer in the space provided. Show work for full credit.

1. Consider the following differential equation for $y(x)$:

$$y'(x) = 4xy - 8x.$$

   (a) (5 points) Find the general solution to this DE, using the algorithm for separable differential equations

Solution: We write this in separable form, and then proceed to separate and integrate. This is a shortcut that is secretly using the chain rule backwards to "integrate" the differential equation. Finally solve the implicit equation for $y(x)$ to explicitly find $y(x)$:

$$\frac{dy}{dx} = 4x(y-2) \Rightarrow \frac{dy}{y-2} = 4xdx$$

$$\Rightarrow \int \frac{dy}{y-2} = \int 4xdx \Rightarrow ln|y-2| = 2x^2 + C_1.$$

Exponentiate, as usual in this situation, and then finish solving for y:

$$|y-2| = e^{2x^2+C_1} = C_2e^{2x^2} \Rightarrow y - 2 = Ce^{2x^2}$$

$$\Rightarrow y = 2 + Ce^{2x^2}.$$

   (b) (5 points) Find the general solution to this DE, using the algorithm for linear differential equations.

Solution: Write this in linear form, compute the integrating factor, multiply both sides by this function, so that you can use the product rule backwards to integrate both sides with respect to x. Finally, divide by the integrating factor to solve for $y(x)$:

$$y'(x) - 4xy(x) = -8x.$$

Integrating factor is $e^{\int P(x)dx} = e^{\int -4xdx} = e^{-2x^2}$ (we choose our favorite antiderivative and don't need the $+C$ at this point), so:

$$e^{-2x^2}(y'(x) + 4xy(x)) = e^{-2x^2}(-8x) \Rightarrow \frac{d}{dx}(e^{-2x^2}y(x) = -8xe^{-2x^2}$$

$$\Rightarrow e^{-2x^2}y = \int -8xe^{-2x^2}dx = 2e^{-2x^2} + C$$
$$\Rightarrow y = 2 + Ce^{2x^2}.$$

2. Consider the following linear drag initial value problem:

$$v'(t) = 6 - 0.2v$$
$$v(0) = 0$$

(a) (3 points) Use a phase diagram to determine the limiting velocity as $t \to \infty$ for the solution to this IVP.

Solution: setting the rate of change function on the right equal to zero yields the equilibrium solution $v \equiv 30$. I prefer to do this by factoring out the leading coefficient, as that usually makes the phase diagram easier to construct:

$$v'(t) = 6 - .02v = -.2(v + \frac{6}{-.02}) = -.2(v - 30).$$

Thus $v < 30 \Rightarrow v'(t) > 0, v > 30 \Rightarrow v'(t) < 0$. Thus the phase diagram is

$$\to\to\to 30 \leftarrow\leftarrow\leftarrow,$$

and the limiting velocity for any initial value v_0 is the terminal velocity $v = 30$.

(b) (4 points) Solve the initial value problem above. (Your solution should be consistent with your phase diagram in part (a).)

Solution: This is a constant coefficient linear DE, and it is also separable. Here is the solution using the linear algorithm:

$$v'(t) + .2v = 6 \Rightarrow e^{.2t}(v'(t) + .2v) = 6e^{.2t}$$
$$\Rightarrow e^{.2t}v = \int 6e^{.2t}dt \Rightarrow e^{.2t}v = \frac{6}{.2}e^{.2t} + C = 30e^{.2t} + C$$
$$\Rightarrow v = 30 + Ce^{-.2t}.$$

Since $v(0) = 0$ we deduce $C = -30$ and

$$v(t) = 30 - 30e^{-.2t}.$$

Thus, as predicted in (a), as $t \to \infty, v(t) \to 30$.

(c) (3 points) Convert the following information into a differential equation initial value problem for the velocity $v(t)$ of a boat moving in a straight line on a lake. Your IVP should end up being equivalent to the IVP at the top of the page: A small motor boat and its pilot weigh a total of 640 pounds. The motor provides a thrust force of 120 pounds. The drag from the boat depends linearly on the velocity v of the boat, and is 4 pounds for each $\frac{ft}{sec}$ of velocity. Aside from the motor and drag forces, there are no other forces acting on the horizontal motion of the boat, and the boat starts from rest. Hint: recall that in the English system, every 32 pounds of force corresponds to 1 slug of mass. (fyi, we use this conversion factor although it is not exact; force in pounds and mass in slugs are related by the gravitational identity $F = mg$, and the acceleration of gravity is not exactly 32 $\frac{ft}{s^2}$.)

Solution: Since the boat plus pilot weight 640 pounds, their total mass is $\frac{640}{32} = 20$ slugs. Newton's second law says

$$mv'(t) = F = F_{thrust} + F_{drag}.$$

$F_{thrust} = 120, F_{drag} = -4v$ are given in the text of the problem. (The minus sign for F_{drag} is because the drag force acts in the opposite direction to the velocity.) Thus

$$20v'(t) = 120 - 4v \Rightarrow v'(t) = 30 - \frac{v}{5}.$$

The initial condition $v(0) = 0$ is given in the text of the problem, since we are told the boat starts from rest.

3. (5 points) Consider a brine tank containing 300 gallons of initially pure water, and with input and output pipes. At time $t = 0$ the system is turned on, with water flow rates of 60 gallons per hour into and out of the tank. The water flowing into the tank is a brine solution, with 0.1 pounds of salt per gallon of brine. Derive the initial value problem for the amount $x(t)$ of salt in the tank at time t. It should be equivalent to the IVP in problem 2. (Since you solved the IVP in problem 2, there is no need to resolve it.)

Solution:

Because the water flow rates (volume per time) are the same coming into and going out of the tank, the volume remains constant, $V = 300$. To find how fast $x(t)$ is changing we use

$$x'(t) = r_{in}c_{in} - r_{out}c_{out} = r_i c_i - r_o c_o.$$

With the well mixed hypothesis, the concentration of the water leaving the tank is the average concentration in the tank, which is $\frac{x(t)}{V}$.. Thus

$$x'(t) = 60(.1) - 60\frac{x}{300} = 6 - .2x.$$

The initial condition $x(0) = 0$ holds, since the water in the tank was "initially pure."