Math 2250-4
Mon Sept 9
1.5: applications of linear DEs, including EP 3.7: electric circuits
Then begin section 2.1, improved population models, if time.

Finish Friday's notes: We discussed the basic input-output model on Friday. If the volume input r_i and output rate r_o are constant, then so is the volume of liquid in the tank. If the incoming concentration c_i is also constant, then the solute amount $x(t)$ will satsify the "constant coefficient" linear DE ($P(t) \equiv a, Q(t) \equiv b$):

$$x' + a\,x = b$$
$$x(0) = x_0$$

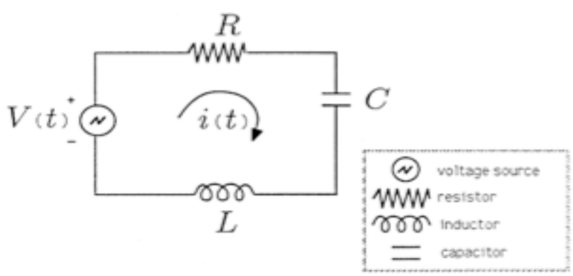
Show this IVP has solution

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}$$

and then doing the application in Friday's notes about a polluted Lake Erie.

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

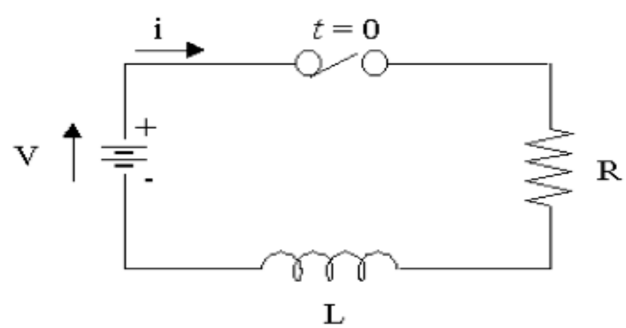
Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.
Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1^{st} order linear DE's, as below:

Exercise 1: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



http://www.intmath.com/differential-equations/5-rl-circuits.php

2.1 Improved population models

Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$) ;

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per $person$)

$D(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{D(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per $person$)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - D(t)$$

$$P'(t) = (\beta(t) - \delta(t))P(t) .$$

<u>Model 1:</u> constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \ge 0$, $\delta(t) \equiv \delta_0 \ge 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = k P .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

<u>Model 2:</u> population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$

$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P))P$$

$$= ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P .$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the <u>logistic differential equation:</u>

$$P' = (b - a P)P$$

$$P' = -a P^2 + b P \text{, or equivalently}$$

$$P' = a P\left(\frac{b}{a} - P\right) = k P(M - P).$$

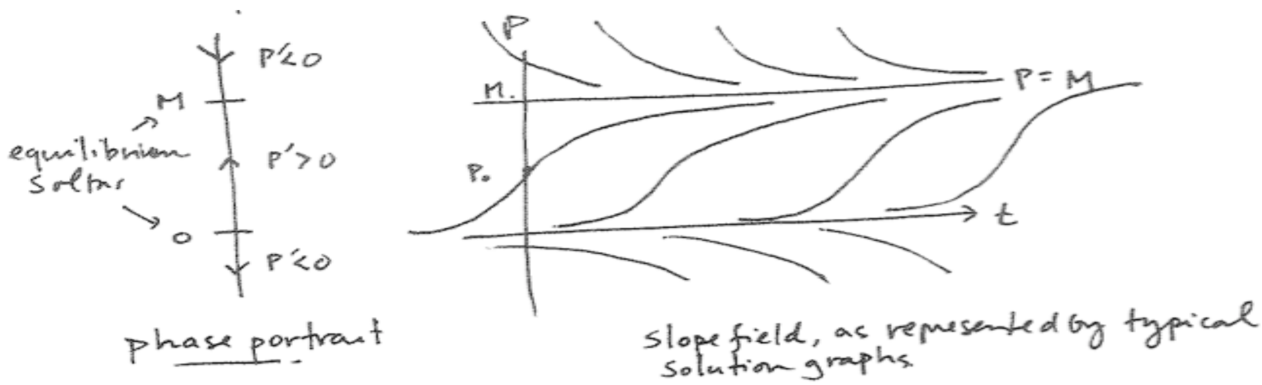
$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 2: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

a) There are two constant ("equilibrium") solutions. What are they?

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram , is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.



c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$?