

Math 2250-4

Fri Sept.6

1.5: linear DEs, and applications.

(If we have time we'll do the Toricelli experiment details from Wednesday's notes; but it's more likely that we'll postpone the experiment for another day.)

Recall the algorithm for solving linear DE's, once they're written in the "linear" form:

$$y' + P(x)y = Q(x)$$

(1) Let $\int P(x)dx$ be any antiderivative of P . Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$e^{\int P(x)dx} (y' + P(x)y) = e^{\int P(x)dx} Q(x)$$

This makes the left-hand side an exact derivative, because of the product rule:

$$\frac{d}{dx} \left(e^{\int P(x)dx} y \right) = e^{\int P(x)dx} Q(x) .$$

(2) So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} Q(x) dx + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

• Remark: If we abbreviate the function $e^{\int P(x)dx}$ by renaming it $G(x)$, then the formula for the solution $y(x)$ to the first order DE above is

$$y(x) = \frac{1}{G(x)} \int e^{\int P(x)dx} Q(x) dx + \frac{C}{G(x)} .$$

If x_0 is a point in any interval I for which the functions $P(x)$, $Q(x)$ are continuous, then $G(x)$ is positive and differentiable, and the formula for $y(x)$ yields a differentiable solution to the DE. By adjusting C to solve the IVP $y(x_0) = y_0$, we get a solution to the DE IVP on the entire interval. And, rewriting the DE as

$$y' = -P(x)y + Q(x)$$

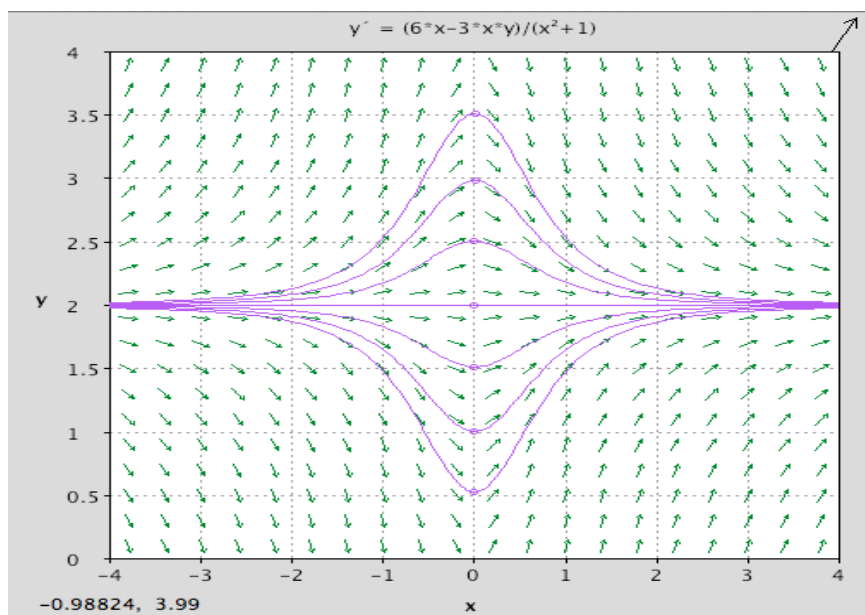
we see that the existence-uniqueness theorem implies this is actually the only solution to the IVP on the interval (since $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y) = P(x)$ are both continuous). These facts would not necessarily be true for separable DE's...and we've seen how separable DE solutions may not exist or be unique on arbitrarily large intervals.

Exercise 1: Find all solutions to the linear (and separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1}$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C(x^2 + 1)^{-\frac{3}{2}}$.

`> with(DEtools) :`
`dsolve(`

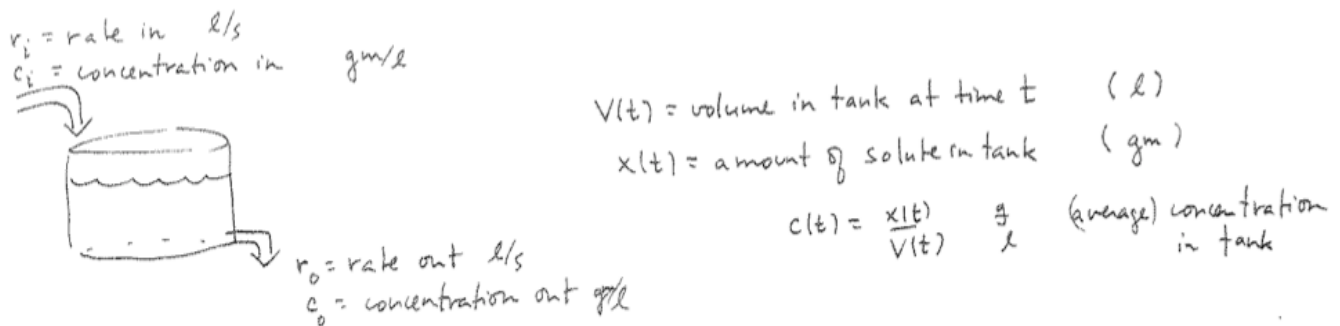


An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_o . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \frac{gm}{l}.$$

See the diagram below.



Exercise 2: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_o$$

so $V(t) = V_0 + \int_0^t (r_i(\tau) - r_o(\tau)) d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$\begin{aligned}x' + a x &= b \\ x(0) &= x_0\end{aligned}$$

where a, b are constants.

Exercise 3: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a} \right) e^{-a t}.$$

Exercise 4: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron has a pretty constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the inflow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?



<http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF>

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$$V = 480 \text{ km}^3$$

$$r = 350 \frac{\text{km}^3}{\text{y}}$$

b) Solve the IVP, and then answer the question.