Math 2250-4
Wed Sept 4
1.4: separable DEs, examples and experiment.
1.5: linear DEs

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week, e.g. exponential growth/decay and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

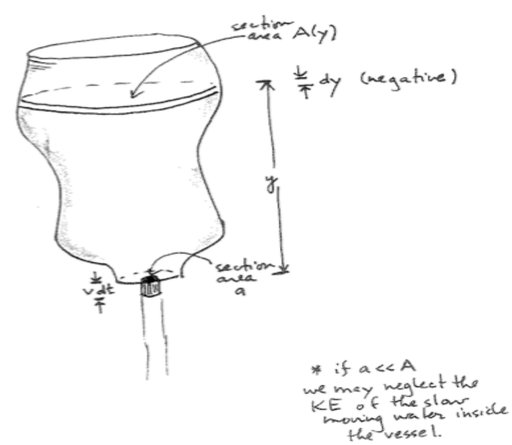
<u>Toricelli's Law</u>, for draining water tanks. Refer to the figure below.
<u>Exercise 1</u>:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity ($a << A$) . Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$v = \sqrt{2 g y}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y) \frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2 g}) .$$



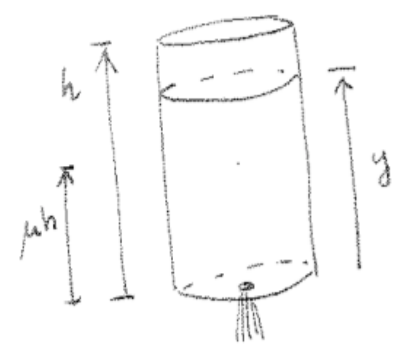
Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k). Let $T(\mu\, h)$ be the time it takes the the water to go from height h (full) to height $\mu \cdot h$, where the fraction μ is between 0 and 1 . Note, $T(h) = 0$ and $T(0)$ is the time it takes for the tank to empty completely.

<u>Exercise 2:</u> (We will use this calculation in our experiment) Solve the IVP and then show the time it takes the tank to empty completely is related to $T(\mu\, h)$ by

$$T(0) = \frac{T(\mu\, h)}{1 - \sqrt{\mu}} .$$



<u>Experiment!</u> We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

> $Digits := 5:$ *# that should be enough significant digits*

> $\dfrac{1}{1 - \text{sqrt}(.5)}$; *# the factor from above, when mu is 0.5*

> $Thalf := 35;$ *# seconds to half-empty canteen*
$Tpredict := 3.4143 \cdot Thalf;$ *#prediction*

>

Remark: Maple can draw direction fields, although they're not as easy to create as in "dfield". On the other hand, Maple can do any undergraduate mathematics computation, including solving pretty much any differential equation that has a closed form solution. Let's see what the commands below produce. We'll get some error messages related to the existence-uniqueness theorem!

```
> with(DEtools) : # this loads a library of DE commands
> deqtn1 := y'(t) = -sqrt(y(t)); # took k=1
  ics1 := y(0) = 1; # took initial height=1
> dsolve({deqtn1, ics1}, y(t)); # DE IVP sol!
> factor(%); #how we wrote it
> DEplot(deqtn1, y(t), 0..3, {[y(0) = 0], [y(0) = 1.], [y(0) = 2.], [y(0) = 0.5], [y(0) = 1.5]},
      arrows = line, color = black, linecolor = black, dirgrid = [30, 30], stepsize = .1, title
      = `Toricelli`);
>
```

....

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x)$$

Exercise 3: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

a) $y'(x) = -2y + 4x^2$

b) $y'(x) = x - y^2 + 1$

c) $y'(x) = x^2 - x^2 y + 1$

d) $y'(x) = \dfrac{6x - 3xy}{x^2 + 1}$

e) $y'(x) = x^2 + y^2$

f) $y'(x) = x^2 e^{x^3}$.

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

$$e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx} Q(x)$$

because the left side is a derivative (product rule):

$$\frac{d}{dx}\left(e^{\int P(x)dx} y\right) = e^{\int P(x)dx} Q(x) .$$

So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} Q(x)dx \quad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

Exercise 4: Solve the differential equation

$$y'(x) = -2\,y + 4\,x^2\,,$$

and compare your solutions to the dfield plot below. You may recognize this DE from your first hw assignment, at which time you didn't know how to derive the solutions: in problem **w1.1** you were told what the solution functions were, and asked to check that they solved the DE.

```
> with(DEtools) :    # load differential equations library
> deqtn2 := y'(x) = -2·y(x) + 4·x^2 :  #notice you must use · for multiplication in Maple,
                                        # and write y(x) rather than y.
  dsolve(deqtn2, y(x)); # Maple check
>
```