Math 2250-4
Wed Sept 25
3.4 Matrix algebra

• Our first exam is next Friday Oct. 4. You have a homework assignment covering 3.5-3.6 which is due a week from today, on Wednesday Oct. 2, and which will be posted on our homework page by later today. (The 3.1-3.4 homework is due this Friday.) The exam will cover through 3.6.

• Today we first finish discussing the general conclusions about how the shape of the reduced row echelon form of a matrix A influences the possible solution sets to linear systems of equations for which it is the coefficient matrix. This is Exercises 5-6 on Tuesday's notes, which is where we ended class yesterday.

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• Then we will discuss vector and matrix algebra, section 3.4.

Matrix vector algebra that we've already touched on, but that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ : \\ x_n + y_n \end{bmatrix} \quad ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ : \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := x_1\,y_1 + x_2\,y_2 + \ldots + x_n\,y_n.$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ : & & & & : \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\,n}x_n \\ : \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\boldsymbol{x}} \\ Row_2(A) \cdot \underline{\boldsymbol{x}} \\ : \\ Row_m(A) \cdot \underline{\boldsymbol{x}} \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\,.$$

Exercise 1a) Compute

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

Exercise 1b): Check that the vector dot product distributes over vector addition and scalar multiplication, i. e.

$$\underline{\boldsymbol{a}} \cdot (\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) = \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{x}} + \underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{y}}$$
$$\underline{\boldsymbol{a}} \cdot (c\, \underline{\boldsymbol{x}}) = c(\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{x}}) .$$

Since the dot product is commutative or by checking directly we also deduce

$$(\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) \cdot \underline{\boldsymbol{a}} = \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{a}} + \underline{\boldsymbol{y}} \cdot \underline{\boldsymbol{a}}$$
$$(c\, \underline{\boldsymbol{a}}) \cdot \underline{\boldsymbol{x}} = c(\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{x}}) .$$

1c) Use your work from b to show that matrix multiplication distributes over vector addition and scalar multiplication, i.e.

$$A(\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}) = A\, \underline{\boldsymbol{x}} + A\, \underline{\boldsymbol{y}}$$
$$A(c\, \underline{\boldsymbol{x}}) = c\, A\, \underline{\boldsymbol{x}}$$

Do this by comparing the i^{th} entries of the vectors on the left, to those on the right. For any vector

$$\underline{\boldsymbol{b}} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}$$

we will use the notation $entry_i(\underline{\boldsymbol{b}})$ for b_i .

Matrix algebra:

- addition and scalar multiplication: Let $A_{m \times n}, B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $entry_{ij}(A) = a_{ij}$, $entry_{ij}(B) = b_{ij}$. (In this case we write $A = [a_{ij}]$, $B = [b_{ij}]$.) Let c be a scalar. Then

$$entry_{ij}(A + B) := a_{ij} + b_{ij}.$$
$$entry_{ij}(c\,A) := c\,a_{ij}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4\,A - B$.

- <u>matrix multiplication:</u> Let $A_{m \times n}$, $B_{n \times p}$ be two matrices such that the number of columns of A equals the number of rows of B. Then the product AB is an $m \times p$ matrix, with

$$entry_{i\,j}(AB) := row_i(A) \bullet col_j(B) = \sum_{k=1}^{n} a_{i\,k} b_{k\,j} \,.$$

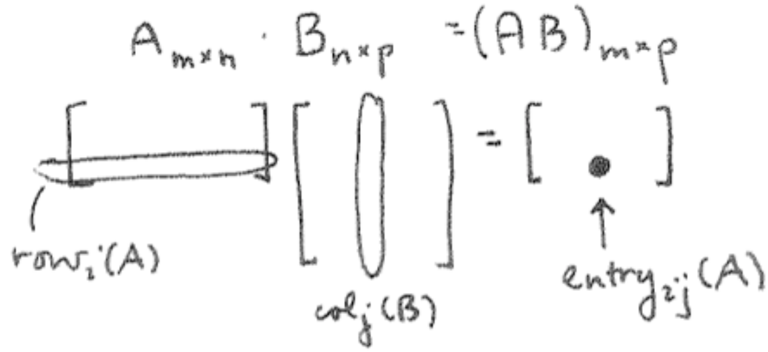
Equivalently, the j^{th} column of AB is given by the matrix times vector product

$$col_j(AB) = A\, col_j(B)$$

and the i^{th} row of AB is given by the product

$$row_i(AB) = row_i(A)\; B.$$

This stencil might help:



<u>Exercise 4)</u>

<u>a)</u> Can you compute AB for the matrices A, B in exercise 3?

b) Let $C := \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Using $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ compute AC and CA and check the row and column properties above. Also notice that $AC \neq CA$, and the sizes of these two product matrices aren't even the same.

Properties for the algebra of matrix addition and multiplication :

- Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $A + B = B + A$
- $+$ is associative $(A + B) + C = A + (B + C)$
- scalar multiplication distributes over $+$ $c(A + B) = cA + cB$.
- multiplication is associative $(AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $A(B + C) = AB + AC$;
  $(A + B)C = AC + BC$

Exercise 5:
a) Verify some of these properties in general - except for the associative property for multiplication they're all easy to check.
b) For the multiplicative associative property verify that at least the dimensions of the triple product matrices are the same.
c) Then check that for the matrices in exercises 3-4, it is indeed true that $(AC)B = A(CB)$.

Identity matrices: The $n \times n$ identity matrix $I_{n \times n}$ has one's down the diagonal (by which we mean the diagonal from the upper left to lower right corner), and zeroes elsewhere. For example,

$$I_{1 \times 1} = [1], \quad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

In other words, $entry_{i\,i}\left(I_{n \times n}\right) = 1$ and $entry_{i\,j}\left(I_{n \times n}\right) = 0$ if $i \neq j$.

Exercise 6) Check that

$$A_{m \times n} I_{n \times n} = A, \qquad I_{m \times m} A_{m \times n} = A .$$

Hint: check that the $i\,j$ entries of each side agree.
(That's why these matrices are called identity matrices - they are the matrix version of multiplicative identities, e.g. like multiplying by the number 1 in the real number system.)

On Friday we will continue our discussion of matrix algebra, focusing on:

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I .$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so

$$B = C.$$

Exercise 7a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Inverse matrices are very useful in solving algebra problems. For example

<u>Theorem:</u> If A^{-1} exists then the only solution to $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

<u>Exercise 7b)</u> Use the theorem and A^{-1} in 7a), to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

<u>Exercise 8)</u> Use matrix algebra to verify why the Theorem is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\,A^{-1}$ (this second product can't even be computed!).