Math 2250-4
Wed Sept 18

- Finish yesterday's discussion of 2.4-2.6, numerical methods for solving first order IVP's. Use yesterday's notes, and the template for Euler, improved Euler, and Runge Kutta, posted on our lecture page. There is also a Matlab file that we might demo.

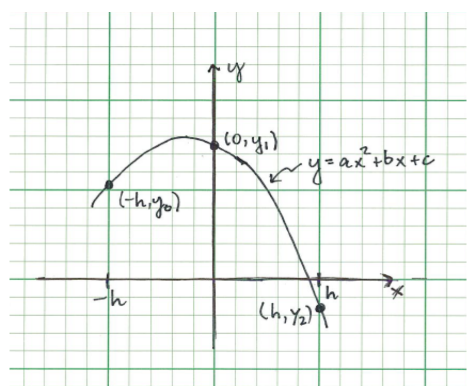
- Use today's notes to introduce Chapter 3, in particular 3.1-3.2 Linear systems of (algebraic) equations and how to solve them...the first exercise relates to this week's homework.

In Chapters 3-4 we temporarily leave differential equations in order to study basic concepts in linear algebra. You've all studied linear systems of equations and matrices before, and that's where Chapter 3 starts. Linear algebra is foundational for many different disciplines, and in this course we'll use the key ideas when we return to higher order linear differential equations and to systems of differential equations. As it turns out, there's an example of solving simultaneous linear equations in this week's homework. It's related to Simpson's rule for numerical integration, which is itself related to the Runge-Kutta algorithm for finding numerical solutions to differential equations.

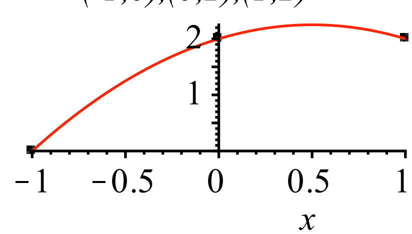
Exercise 1: Set up a system of three linear algebraic equations for the coefficients a, b, c of a quadratic function $p(x) = a\,x^2 + b\,x + c$ so that for some fixed $h > 0$ the graph $y = p(x)$ goes through the three points

$$(-h, y_0),\ (0, y_1),\ (h, y_2)$$

(See example of what you're trying to do, below....this problem relates to Simpson's rule for numerical integration, see discussion in hw.)



For example, the graph of $p(x) = -x^2 + x + 2$ interpolates the three points $(-1, 0)$, $(0, 2)$, $(1, 2)$:



In 3.1-3.2 our goal is to understand systematic ways to solve simultaneous linear equations. Although we used a, b, c for the unknowns in the previous problem, this is not our standard way of labeling.

- We'll often call the unknowns $x_1, x_2, \ldots x_n$, or write them as elements in a vector

$$\underline{\mathbf{x}} = \left[x_1, x_2, \ldots x_n\right].$$

- Then the general linear system (LS) of m equations in the n unknowns can be written as

$$\begin{array}{l} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad\quad \vdots \quad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n = b_m \end{array}$$

where the coefficients $a_{i\,j}$ and the right-side number b_j are known. The goal is to find values for the vector $\underline{\mathbf{x}}$ so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once.)

<u>Notice</u> that we use two subscripts for the coefficients $a_{i\,j}$ and that the first one indicates which equation it appears in, and the second one indicates which variable its multiplying; in the corresponding *coefficient matrix A* , this numbering corresponds to the row and column of $a_{i\,j}$:

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} \end{bmatrix}$$

Let's start small, where geometric reasoning will help us understand what's going on:

<u>Exercise 2</u>: Describe the solution set of each single equation below; describe and sketch its geometric realization in the indicated Euclidean spaces.

<u>2a)</u> $3\,x = 5$, for $x \in \mathbb{R}$.

<u>2b)</u> $2\,x + 3\,y = 6$, for $[x, y] \in \mathbb{R}^2$.

<u>2c)</u> $2\,x + 3\,y + 4\,z = 12$, for $[x, y, z] \in \mathbb{R}^3$.

2 linear equations in 2 unknowns:

$$a_{11}\, x + a_{12}\, y = b_1$$
$$a_{21}\, x + a_{22}\, y = b_2$$

goal: find all $[x, y]$ making both of these equations true. So geometrically you can interpret this problem as looking for the intersection of two lines.

Exercise 3: Consider the system of two equations E_1, E_2:

$$E_1 \qquad 5\,x + 3\,y = 1$$
$$E_2 \qquad x - 2\,y = 8$$

3a) Sketch the solution set in $\mathbb{R}^2$, as the point of intersection between two lines.
3b) Use the following three "elementary equation operations" to systematically reduce the system E_1, E_2 to an equivalent system (i.e. one that has the same solution set), but of the form

$$1\,x + 0\,y = c_1$$
$$0\,x + 1\,y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

3c) Look at your work in 3b. Notice that you could have save a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using R_1, R_2 as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

	5	3	1		
	1	-2	8		
R_2	1	-2	8		
R_1	5	3	1		
	1	-2	8		
$-5R_1+R_2$	0	13	-39		
	1	-2	8		
	0	1	-3		
$2R_2+R_1$	1	0	2	$\rightarrow$	$x = 2$!
	0	1	-3		$y = -3$

3d) What are the possible geometric solutions sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

Solutions to linear equations in 3 unknowns:

What is the geometric question you're answering?

Exercise 4) Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23 .$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about. It's called Gaussian elimination.

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

Exercise 5 There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 4, except for a_{33} , and we changed the right side in the third equation, for 4a. Work out what happens in each case.

5a)

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 8z = 20 .$$

5b)

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 8z = 23 .$$

5c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?