Math 2250-4
Wed Sept 11
2.2 Autonomous differential equations

- Complete the discussion in yesterday's notes about the following important concepts:
  * Autonomous first order DE
  * equilibrium solutions for autonomous DE's
  * stability at equilibrium points.
- Then continue with today's notes.

Exercise 1) Use the chain rule to check that if $x(t)$ solves the autonomous DE
$$x'(t) = f(x)$$
Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic: $\qquad\qquad P'(t) = -a P^2 + b P$

Doomsday-extinction: $\qquad Q'(t) = a Q^2 - b Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators), i.e. the fertility rate $\beta = a Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a Q^2$ and $-b Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a Q\left(Q - \frac{b}{a}\right) = k Q(Q - M).$$

Exercise 2a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

Exercise 2b) If $P(t)$ solves the logistic differential equation
$$P'(t) = k P(M - P)$$
show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation
$$Q'(t) = k Q(Q - M) .$$
Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{M P_0}{(M - P_0)e^{-Mkt} + P_0} .$$

Exercise 3: Use your formula from the previous exercise to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x-1)$$
$$x(0) = 2.$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

$x' = x*(x-1)$ Arrow of slope $+1.0$

0.1986, 2.9129
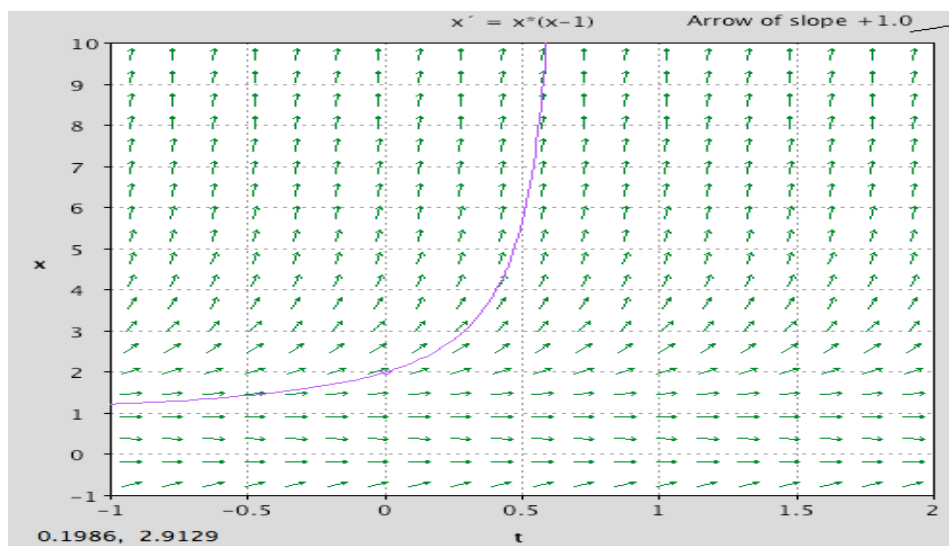
harvesting a logistic population...text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?) Consider the DE
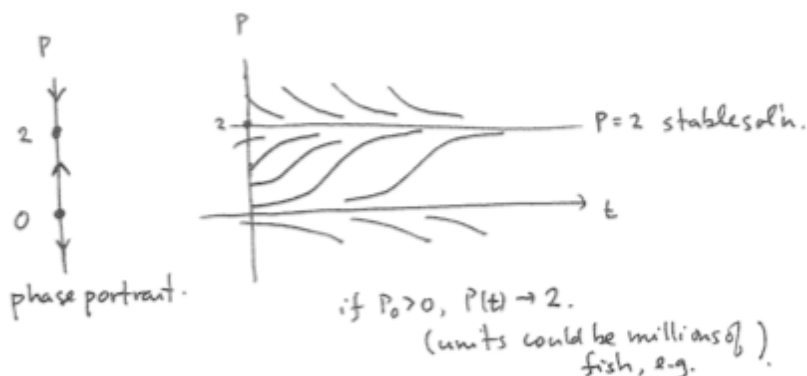
$$P'(t) = a P - b P^2 - h .$$

Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h P$ instead of P .

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity
take $a = 2$, $b = 1$

Case 0 no harvesting

$$P'(t) = 2P - P^2$$
$$= P(2 - P)$$

phase portrait.
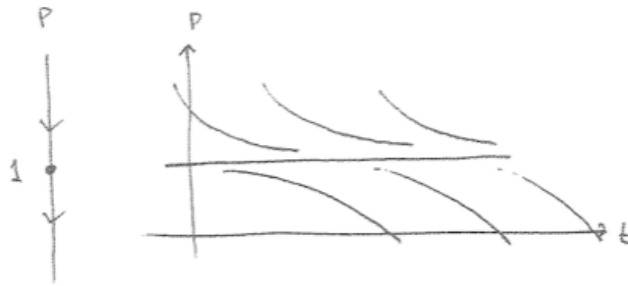
$P = 2$ stable sol'n.

if $P_0 > 0$, $P(t) \to 2$.
(units could be millions of fish, e.g.)

with harvesting:

$$P'(t) = 2P - P^2 - h$$
$$= -(P^2 - 2P + h)$$
$$= -(P - P_1)(P - P_2)$$

$$P_1, P_2 = \frac{2 \pm \sqrt{4 - 4h}}{2}$$
$$= 1 \pm \sqrt{1 - h}$$

Case 1: subcritical harvesting

$$0 < h < 1$$

$1 + \sqrt{1-h}$

$1 - \sqrt{1-h}$

$P = 1 + \sqrt{1-h}$

$P = 1 - \sqrt{1-h}$

<u>Case 2.</u> Critical harvesting

$$h = 1$$

$$P'(t) = -(P-1)^2$$

1

P   P

<u>Case 3</u> Over harvesting

$$h > 1$$

complex roots.

$$P'(t) = -(P^2 - 2P + h)$$
$$= -[(P-1)^2 + (h-1)]$$
$$< 0.$$

P
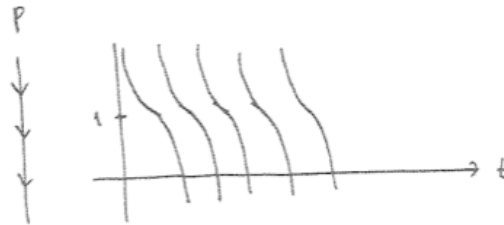
This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \to 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1 - h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c\,(2 - c)$.

"bifurcation diagram" of
equilibrium sol'ns in the h-c
plane

c (equil sol's.