Math 2250-4
Tues Sept 10
2.1-2.2 Improved population models; equilibria and stability for autonomous first order DE's.

Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$$\beta(t) := \frac{B(t)}{P(t)}$$, fertility rate ($\frac{people}{year}$ per $person$)

$D(t)$, death rate (e.g. $\frac{people}{year}$);

$$\delta(t) := \frac{D(t)}{P(t)}$$, mortality rate ($\frac{people}{year}$ per $person$)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - D(t)$$
$$P'(t) = (\beta(t) - \delta(t))P(t) .$$

<u>Model 1:</u> constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = k\,P .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

<u>Model 2:</u> population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \left((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P)\right)P$$
$$= \left((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P\right)P .$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the <u>logistic differential equation:</u>

$$P' = (b - a\,P)P$$
$$P' = -a\,P^2 + b\,P$$, or equivalently
$$P' = a\,P\left(\frac{b}{a} - P\right) = k\,P(M - P).$$

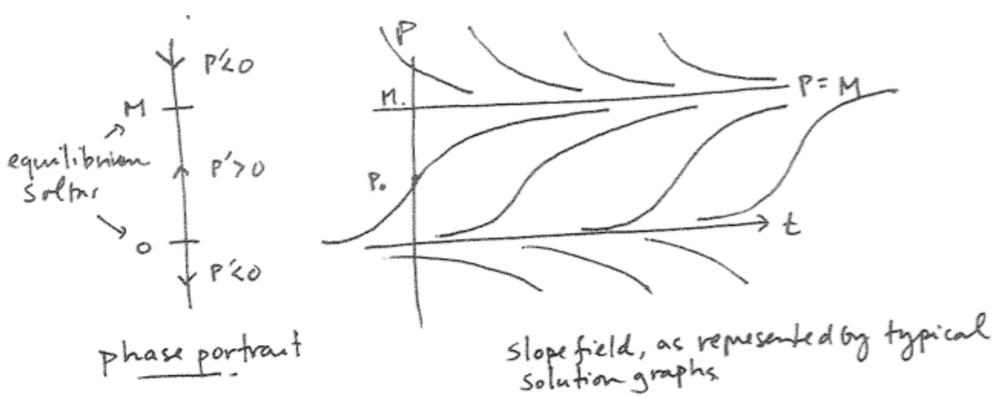
$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

a) There are two constant ("equilibrium") solutions. What are they?

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram , is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.



c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M . Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$?

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}\ .$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P - M)} = -k\,dt$$

By partial fractions,

$$\frac{1}{P(P - M)} = \frac{1}{M}\left(\frac{1}{P - M} - \frac{1}{P}\right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P - M} - \frac{1}{P}\right)dP = -kM\,dt\,.$$

Integrate:

$$\ln|P - M| - \ln|P| = -Mkt + C_1$$
$$\ln\left|\frac{P - M}{P}\right| = -Mkt + C_1$$

exponentiate:

$$\left|\frac{P - M}{P}\right| = C_2\,e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P - M}{P} = C\,e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C\,.)$$

Now, solve for $P(t)$ by multiplying both sides of of the second to last equation by $P(t)$:

$$P - M = Ce^{-Mkt}P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$P - Ce^{-Mkt}P = M$$
$$P\left(1 - Ce^{-Mkt}\right) = M$$
$$P = \frac{M}{1 - Ce^{-Mkt}}\ .$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\frac{P_0 - M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} .$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

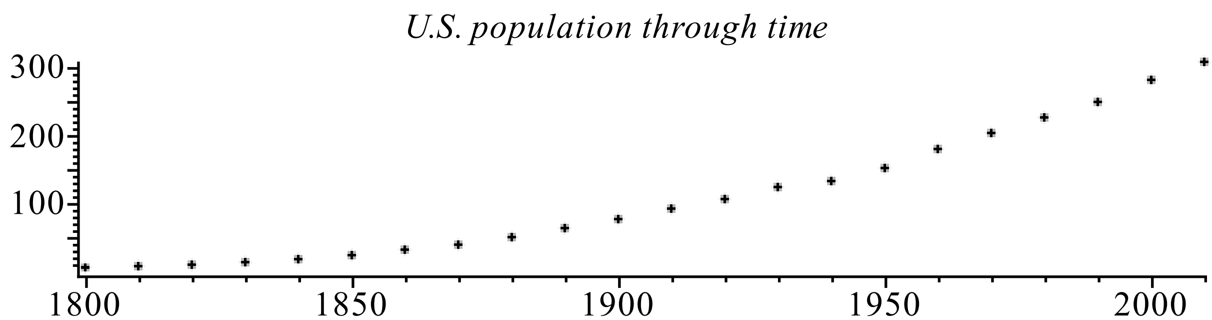
$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

<u>Note:</u> If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```



Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \quad (1)$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \to 5.308\, e^{0.02664\, t} \quad (2)$$

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\,P0}{P0 + (M - P0)\,e^{-M k t}} \tag{3}$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \tag{4}$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
     #equation (11) page 84.
```

$$M := 188.12$$

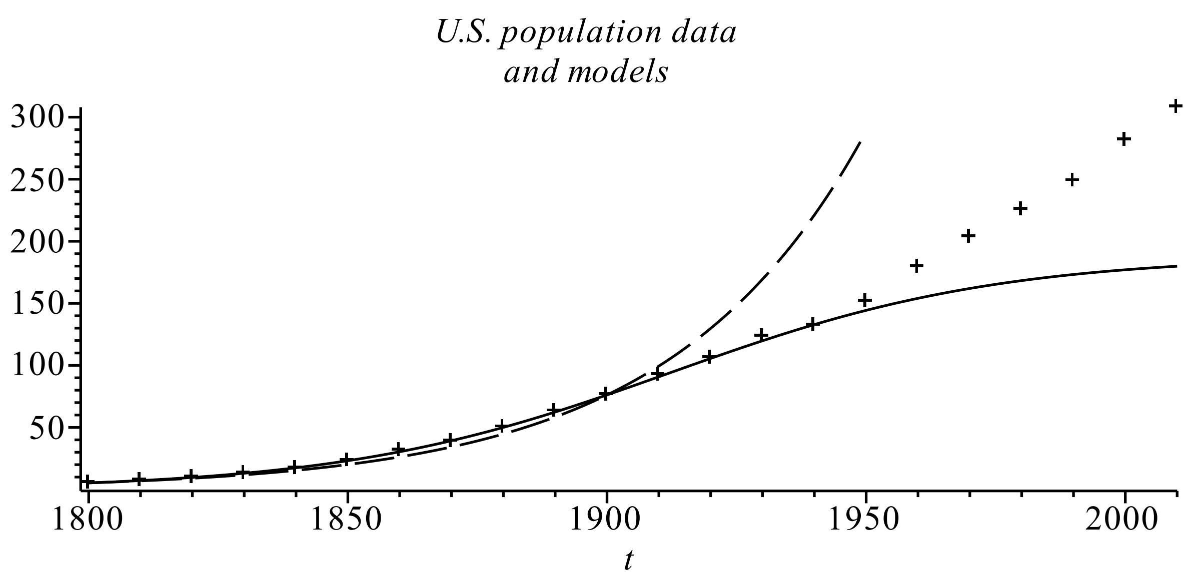
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\,e^{-0.031551\,t}} \tag{5}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t−1800), t = 1800..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t−1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
```

2.2: Autonomous Differential Equations.

Recall, that if we solve for the derivative, a general first order DE for $x = x(t)$ is written as

$$x' = f(t, x) ,$$

which is shorthand for $x'(t) = f(t, x(t))$.

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) .$$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$, for example.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 3: Find the equilibrium solutions of

3a) $x'(t) = 3x - x^2$

3b) $x'(t) = x^3 + 2x^2 + x$

3c) $x'(t) = \sin(x)$.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

Exercise 4: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 3. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

4a) $x'(t) = 3x - x^2$

4b) $x'(t) = x^3 + 2x^2 + x$

4c) $x'(t) = \sin(x)$.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

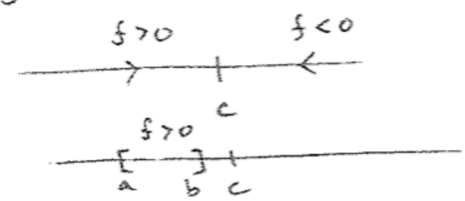
with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f)$: - - - - 0 + + + $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable
$sign(f)$: + + + 0 - - - - $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is asymptotically stable
$sign(f)$: + + + 0 + + + $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable (half stable)
$sign(f)$: - - - - 0 - - - - $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum).

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ■