Math 2250-4
Wed Nov 6

5.6 continued: Forced mechanical (and electrical) oscillations.

Yesterday we started to discuss the physical phenomena which arise in undamped forced oscillation problems, and the mathematics that explains these phenomena:

$$m\,x'' + k\,x = F_0\cos(\omega\,t)$$

$$x'' + \frac{k}{m}\,x = \frac{F_0}{m}\cos(\omega\,t)$$

$$x'' + \omega_0^2\,x = \frac{F_0}{m}\cos(\omega\,t)$$

We used section 5.5 undetermined coefficients algorithms. The solutions are:

- $\omega \neq \omega_0$ undetermined coefficients implies

$$x(t) = x_P + x_H = A\cos(\omega\,t) + C_0\cos(\omega_0 t - \alpha)$$

which may or may not be periodic, depending on whether the two sinusoidal functions have a common period.

After calculation, one verifies that for initial conditions

$$x(0) = x_0,\ x'(0) = v_0$$

the solution is

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}\left(\cos(\omega\,t) - \cos(\omega_0 t)\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

(That's as far as we got on Tuesday's notes, which we will return to, to finish the rest of the undamped-forced discussion summarized below.)

If $\omega \approx \omega_0$ but $\omega \neq \omega_0$ beating will occur as the difference of cosines above is sometimes in-phase and sometimes out of phase. We can study this more carefully by using cosine angle addition formulas to rewrite the IVP solution above as

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

- $\omega = \omega_0$ Resonance: undetermined coefficients implies there is a particular solution

$$x_P = t\left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)\ .$$

After calculation, one verifies that the IVP solution is

$$x(t) = \frac{F_0}{2\,m\,\omega_0}t\sin(\omega_0\,t) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

• After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)\ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Damped forced oscillations ($c > 0$) for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega\,t) + B\sin(\omega\,t)\right] \\ +\,c\,&\left[x_P' = -A\,\omega\sin(\omega\,t) + B\,\omega\cos(\omega\,t)\right] \\ +\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega\,t) - B\,\omega^2\sin(\omega\,t)\right]\,. \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega\,t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\ &\quad + \sin(\omega\,t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right)\,. \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\,. \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\,.$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)\,.$
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 0) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0, \cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0, \cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1, \cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 1) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3).$$

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3e^{-t}\sin(5t) + e^{-t}\cos(5t) + 5\cos(4t) + 4\sin(4t) \quad (1)$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations (page 3) is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 2a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

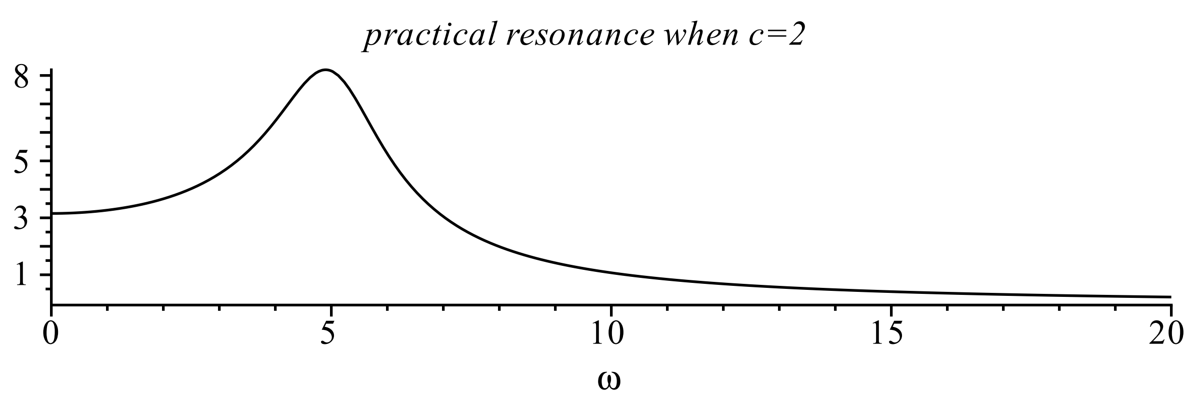
$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

2b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

```
> with(plots):
```

$$> C := (\omega, c) \to \frac{82}{\sqrt{\left(26 - \omega^2\right)^2 + c^2\cdot\omega^2}} :$$

```
> plot(C(ω, 2), ω = 0..20, color = black, title = `practical resonance when c=2`);
```



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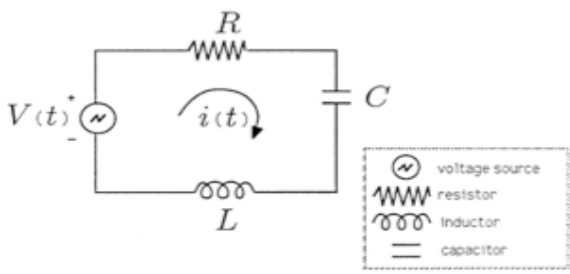
The mechanical-electrical analogy, continued: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega\, t)$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\, \omega \cos(\omega\, t)$.

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t) .$$

$$I_{sp}(t) = I_0 \cos(\omega\, t - \alpha) = I_0 \sin(\omega\, t - \gamma)\ ,\ \ \gamma = \alpha - \frac{\pi}{2} .$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}} .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

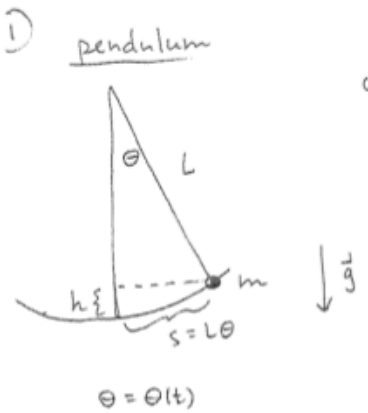
$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:
http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

If you're an engineer concerned about resonance or practical resonance, you need to know how to deduce the natural frequencies of undamped mechanical (or electrical) systems. Usually the best way is to use conservation of energy in mechanical systems, which is an integrated and more generally applicable version of Newton's second law for mechanical systems (when energy is actually conserved). Conservation of potential energy around closed loops for undamped electrical circuits *is* Kirchoff's law.

Here are two examples, one old and one new:

- We've carefully discussed the (linearized) pendulum model, which leads to $\omega_0 = \sqrt{\frac{g}{L}}$:



conservative system $KE + PE = \text{const.}$

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}mL^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}(L\theta'' + g\sin\theta) \equiv 0$

⤳ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize ↙ $\qquad$ ↘ non-linear DE

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

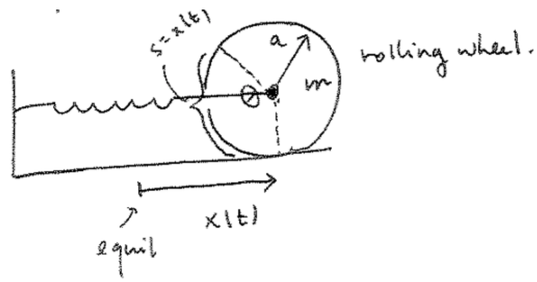
$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\sin\theta \approx \theta$ $\quad \theta$ small is excellent approx (alternating series test)

Exercise 3) Multicomponent systems are best understood using conservation of energy, when Newton's law may not apply in any obvious way. For example, consider the following "rolling mass" configuration (the spring constant of the massless spring is not shown, but as usual we call it k.)



Find the natural angular frequency for the configuration above. Do this by first computing $TE = KE + PE$. Then express the fact that TE is constant once the system is set in motion, by computing its time derivative and setting it equal to the zero function. Use:

(1) The potential energy from stretching/compressing a spring from equilbrium $x = 0$ is the work done in moving it from equilbrium to displacement x, i.e.

$$PE = W = \int_0^x k\, s \; ds = \frac{1}{2} k\, x^2 .$$

(2) The KE_{trans} from the linear motion (translation) of the disk is $\frac{1}{2} m\, x'(t)^2$, since $x(t)$ is tracking the center of mass' displacement from equilbrium

(3) The KE_{rot} from the rotation of the disk is given by

$$KE_{rot} = \frac{1}{2} I\, \omega^2 ,$$

where $\omega = \theta'(t)$ is the angular frequency of the rotation and I is the moment of inertia, which for a uniform disk of mass m and radius a is given by $I = \frac{1}{2} m a^2$. (Directly computing this rotational KE is a double integral computation, that works out nicely using polar coordinates. You might even have done it in your multivariable calculus class if and when you discussed moments of intertia....in general, moments of intertia are used to compute rotational kinetic energy about centers of mass, and moments are used to compute angular momentum, as well as centers of mass....this is why Calculus classes have units about these topics.)

(The answer is $\omega_0 = \sqrt{\frac{2}{3}} \sqrt{\frac{k}{m}} \approx .82 \sqrt{\frac{k}{m}}$, which is slower than if the mass wasn't rolling.)