Math 2250-4
Tues Nov 5

• Finish section 5.5 undetermined coefficients (yesterday's notes, "extended case"). The more general but usually harder variation of parameters method for finding particular solutions is at the end of today's notes, and we may use it to do one of the exercises today. After we finish yesterday's notes, begin

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on Wednesday. Today we will focus on undamped forced oscillations, and tomorrow we'll focus on the damped case.

Overview for solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 5.5 undetermined coefficients algorithms.

• undamped $(c = 0)$:
In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

which has simple harmonic motion solutions $x_H(t) = C\cos(\omega_0 t - \alpha)$. So for the non-homongeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

• $\omega \neq \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need $\sin(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

• $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!

• $\omega = \omega_0$ $\Rightarrow x_P = t\left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

• damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm\,i\,\omega$ when $c > 0$).

• underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)\,.$
• critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
• over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

- in all three cases on the previous page, $x_H(t) \to 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important.

<u>forced undamped oscillations</u>:

<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

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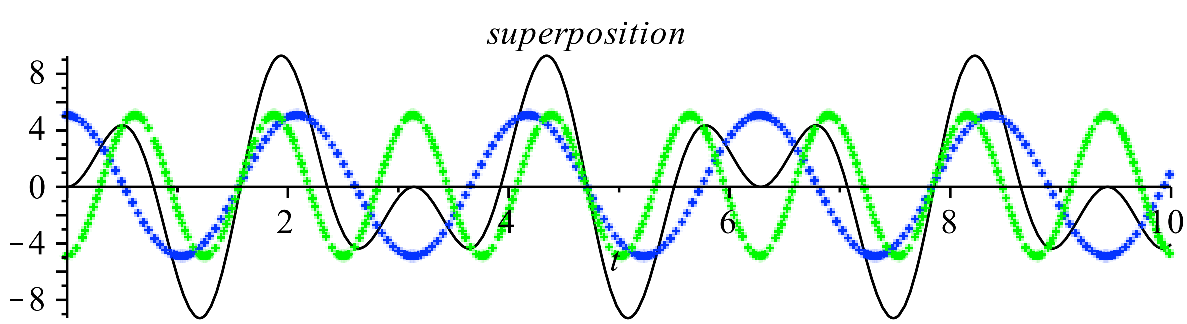
> with(plots) :
> plot1 := plot(−5·cos(5·t), t = 0 ..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0 ..10, color = blue, style = point) :
  plot3 := plot(−5·cos(5·t) + 5·cos(3·t), t = 0 ..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned} +\tfrac{k}{m} & \left(x_p = A \cos \omega t \right) \\ +0 & \left(x_p' = -A\omega \sin \omega t \right) \\ +1 & \left(x_p'' = -A\omega^2 \cos \omega t \right) \\ \hline L(x_p) & = \cos \omega t \, A \left[\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2 \right] \end{aligned}$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$$A = \frac{F_0}{m} \Big/ (\omega_0^2 - \omega^2)$$

So,

$x_p(t) = -\frac{F_0}{m} \frac{1}{(\omega^2 - \omega_0^2)} \cos \omega t$. Note $x_H(t) = A \cos \omega_0 t + B \sin \omega_0 t$.

so, by plugging in or observation

IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2 - \omega_0^2)} \left(\cos \omega_0 t - \cos \omega t \right) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t}$$

check-WR!

There is an interesting <u>beating</u> phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned} \cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - \left(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \right) \\ &= 2 \sin(\alpha)\sin(\beta) \ . \end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t \right) \sin\left(\frac{1}{2}(\omega - \omega_0)t \right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|} \ .$$

We will call half that period the beating period, as explained by the next exercise:

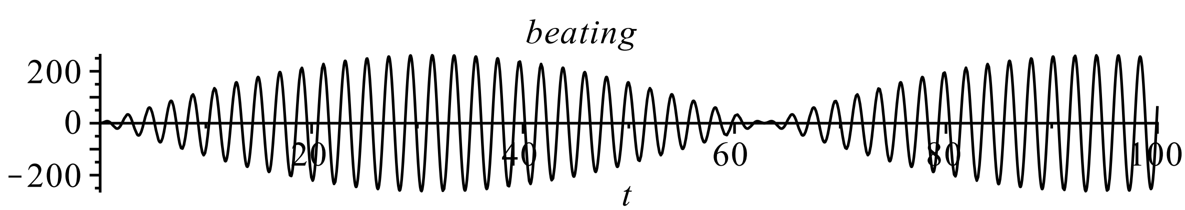
$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2-\omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

> $plot(262.3\cdot\sin(3.05\cdot t)\sin(.05\cdot t),\ t=0..100,\ color=black,\ title=\text{`beating`});$



Resonance:

Resonance! $\omega=\omega_0$ (and the limit as $\omega\to\omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using § 5.5, guess

$$\begin{aligned}
+\omega_0^2(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) &)\\
0\,(\quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t &)\\
+1\,(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + [-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t]\,2 &)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\,[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A=0$, $B=\frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

(You can also get this solution by letting $\omega\to\omega_0$ in the beating formula.)

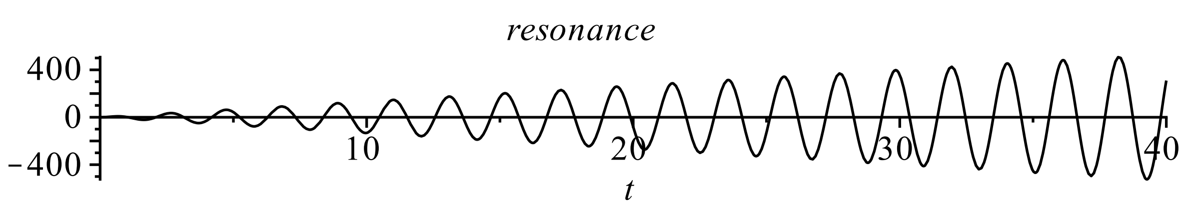
Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot t),\ t = 0..40,\ color = black,\ title = `resonance`\right);$



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Variation of Parameters: This is an alternate method for finding particular solutions. Its advantage is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 .$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \left[W\left(y_1, y_2, \ldots, y_n\right)\right]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

where $\left[W\left(y_1, y_2, \ldots, y_n\right)\right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1 y_1 + u_2 y_2 .$$

Thus

$$y' = u_1 y_1' + u_2 y_2' + \left(u_1' y_1 + u_2' y_2\right) .$$

Set

$$\left(u_1' y_1 + u_2' y_2\right) = 0.$$

Then

$$y'' = u_1 y_1'' + u_2 y_2'' + \left(u_1' y_1' + u_2' y_2'\right) .$$

Set

$$\left(u_1' y_1' + u_2' y_2'\right) = f.$$

Notice that the two $(\ldots)$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$\begin{aligned} & p_0\left[y = u_1 y_1 + u_2 y_2\right] \\ & + p_1\left[y' = u_1 y_1' + u_2 y_2'\right] \\ & + 1\left[y'' = u_1 y_1'' + u_2 y_2'' + f\right] \\ & \quad L(y) = u_1 L\left(y_1\right) + u_2 L\left(y_2\right) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$