

Math 2250-4
Tues Nov 26

Continue discussing systems of differential equations, 7.1-7.3

- Finish exercises 3, 4 in Monday's notes, which foreshadow the eigenvalue-eigenvector method for solving homogeneous first order systems of differential equation IVPs

$$\begin{aligned}\underline{x}'(t) &= A \underline{x} \\ \underline{x}(t_0) &= \underline{x}_0\end{aligned}$$

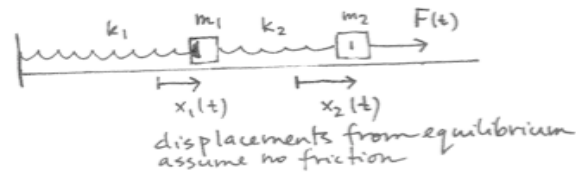
that the book covers in detail in section 7.3.

- Then discuss the important fact that every n^{th} order differential equation or system of differential equations is actually equivalent to a (possibly quite large) system of first order differential equations. This discussion is in today's notes and is the content of section 7.1. The existence-uniqueness theorem for first order systems of differential equations will then explain why the natural initial value problems for higher order differential equations, that we studied in Chapter 5, also always have unique solutions. Another mystery that this correspondence will solve is why we used "characteristic polynomial" when looking for basis solutions e^{rt} to n^{th} order constant coefficient homogeneous differential equations and then used exactly the same terminology, "characteristic polynomial", when we were looking for matrix eigenvalues (see section 7.3).
- Along the way and tomorrow as well, we will also be discussing how the theory and template for finding solutions to first order systems of linear differential equations precisely mirrors the template we developed for single n^{th} -order linear differential equations in Chapter 5. This is the content of section 7.2.

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

Example:

Consider this configuration of two coupled masses and springs:



Exercise 1) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 2) Consider the IVP from Exercise 1, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40 \sin(3t)$:

$$\begin{aligned}x_1'' &= -3x_1 + x_2 \\x_2'' &= 2x_1 - 2x_2 + 40 \sin(3t) \\x_1(0) &= b_1, x_1'(0) = b_2 \\x_2(0) &= c_1, x_2'(0) = c_2.\end{aligned}$$

2a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}v_1(t) &:= x_1'(t) \\v_2(t) &:= x_2'(t)\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}x_1' &= v_1 \\x_2' &= v_2 \\v_1' &= -3x_1 + x_2 \\v_2' &= 2x_1 - 2x_2 + 40 \sin(3t) \\x_1(0) &= b_1 \\v_1(0) &= b_2 \\x_2(0) &= c_1 \\v_2(0) &= c_2.\end{aligned}$$

2b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

It is always the case that the natural initial value problems for single or systems of differential equations are equivalent to an initial value problem for a larger system of first order differential equations, as in the previous example.

A special case of this fact is that if you have an IVP for single n^{th} order DE for $x(t)$, it is equivalent to an IVP for a system of n first-order DE's, in which the functions are

$$x_1(t) = x(t), \quad x_2(t) := x'(t), \quad x_3(t) := x''(t), \dots, \quad x_n(t) := x^{(n-1)}(t).$$

For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$

$$x(0) = 4$$

$$x'(0) = -4.$$

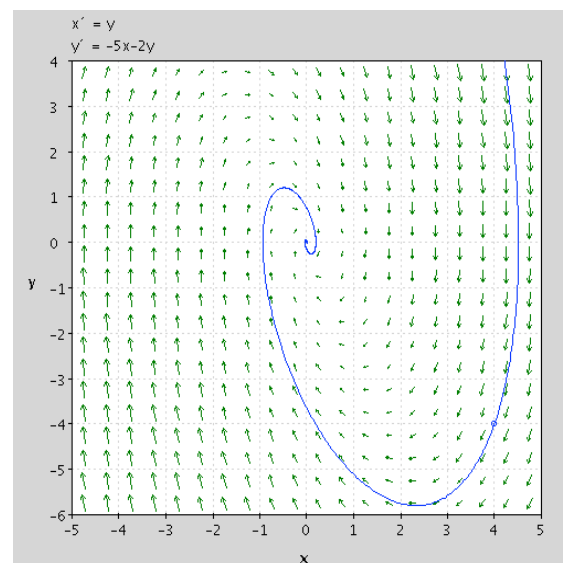
Exercise 3)

3a) Convert this single second order IVP into an equivalent first order system IVP for $x_1(t) := x(t)$ and $x_2(t) := x'(t)$.

3b) Solve the second order IVP in order to deduce a solution to the first order IVP. Use Chapter 5 methods even though you love Laplace transform more.

3c) How does the Chapter 5 "characteristic polynomial" in 3b compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 3a? hmmm.

3d) Is your analytic solution $[x(t), v(t)]$ in 3b consistent with the parametric curve shown below, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.



Theorem 1 For the IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If $\underline{\mathbf{F}}(t, \underline{\mathbf{x}})$ is continuous in the t -variable and differentiable in its $\underline{\mathbf{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= A(t)\underline{\mathbf{x}}(t) + \underline{\mathbf{f}}(t) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If the matrix $A(t)$ and the vector function $\underline{\mathbf{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commercial numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$\begin{aligned}t_j &= t_0 + hj \\ \underline{\mathbf{x}}_{j+1} &= \underline{\mathbf{x}}_j + h \underline{\mathbf{F}}(t_j, \underline{\mathbf{x}}_j) .\end{aligned}$$

Remark: These theorems are the true explanation for why the n^{th} -order linear DE IVPs in Chapter 5 always have unique solutions - because each n^{th} - order linear DE IVP is equivalent to an IVP for a first order system of n linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{x}(t)$ differentiable on an interval, the operator

$$L(\underline{x}(t)) := \underline{x}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$\begin{aligned} L(\underline{x}(t) + \underline{z}(t)) &= L(\underline{x}(t)) + L(\underline{z}(t)) \\ L(c \underline{x}(t)) &= c L(\underline{x}(t)) . \end{aligned}$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{f}(t)$$

$\forall t \in I$ is

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$$

where $\underline{x}_p(t)$ is any single particular solution and $\underline{x}_H(t)$ is the general solution to the homogeneous problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{0}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{x}'(t) = A(t)\underline{x}(t) .$$

3.3) For $A(t)_{n \times n}$ and $\underline{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\underline{x}' = A \underline{x}$$

is n -dimensional. Here's why:

- Let $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$[W(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n)](t_0) := [\underline{X}_1(t_0) | \underline{X}_2(t_0) | \dots | \underline{X}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{b} \in \mathbb{R}^n$ the IVP

$$\begin{aligned} \underline{x}' &= A \underline{x} \\ \underline{x}(t_0) &= \underline{b} \end{aligned}$$

has solution $\underline{x}(t) = c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t) + \dots + c_n \underline{X}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\begin{bmatrix} \underline{X}_1(t_0) & \underline{X}_2(t_0) & \dots & \underline{X}_n(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$, and since each IVP has only one solution, $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{b} = \underline{0}$) is the one with $\underline{c} = \underline{0}$. Thus $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

7.3 Eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\mathbf{x}}' = A \underline{\mathbf{x}}$$

Here's how: We look for a basis of solutions $\underline{\mathbf{x}}(t) = e^{\lambda t} \underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ is a constant vector. Substituting this form of potential solution into the system of DE's above yields the equation

$$\lambda e^{\lambda t} \underline{\mathbf{v}} = A e^{\lambda t} \underline{\mathbf{v}} = e^{\lambda t} A \underline{\mathbf{v}}.$$

Dividing both sides of this equation by the scalar function $e^{\lambda t}$ gives the condition

$$\lambda \underline{\mathbf{v}} = A \underline{\mathbf{v}}.$$

- We get a solution every time $\underline{\mathbf{v}}$ is an eigenvector of A with eigenvalue λ !
- If A is diagonalizable then there is an $\mathbb{R}^n$ basis of eigenvectors $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_n$ and solutions

$$\underline{\mathbf{X}}_1(t) = e^{\lambda_1 t} \underline{\mathbf{v}}_1, \underline{\mathbf{X}}_2(t) = e^{\lambda_2 t} \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{X}}_n(t) = e^{\lambda_n t} \underline{\mathbf{v}}_n$$

which are a basis for the solution space on the interval $I = \mathbb{R}$, because the Wronskian matrix at $t = 0$ is the invertible diagonalizing matrix

$$P = [\underline{\mathbf{v}}_1 | \underline{\mathbf{v}}_2 | \dots | \underline{\mathbf{v}}_n]$$

that we considered in Chapter 6.

- If A has complex number eigenvalues and eigenvectors it may still be diagonalizable over $\mathbb{C}^n$, and we will still be able to extract a basis of real vector function solutions. If A is not diagonalizable over $\mathbb{R}^n$ or over $\mathbb{C}^n$ the situation is more complicated.

Exercise 4a) Use the method above to find the general homogeneous solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4b) Solve the IVP with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Exercise 5a) What second order overdamped initial value problem is equivalent to the first order system IVP on the previous page. And what is the solution function to this IVP?

5b) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE in 4a, and the Chapter 7 "Wronskian matrix" for the solution to the equivalent first order system?

5c) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Interpret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

5d) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \rightarrow \infty$, and the direction solutions came from (as $t \rightarrow -\infty$)?

<http://math.rice.edu/~dfield/dfpp.html>

