Math 2250-4
Mon Nov 25

• Finish last Friday's discussion of convolution solutions to inhomogeneous linear constant coefficient differential equations.

• Then begin:

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:
http://en.wikipedia.org/wiki/Multi-compartment_model

Here's a relatively simple 2-tank problem to illustrate the ideas:

$V_1 = 100g$ $V_2 = 200g$
$x_1(t)$ $x_2(t)$
solute. $r = 400 \, g/h$
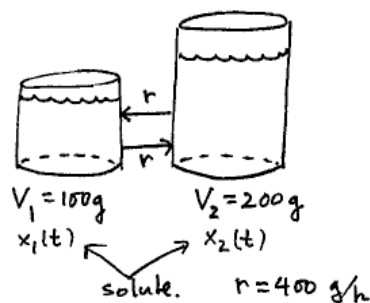
Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

The example on page 1 is a special case of the general <u>initial value problem for a first order system of</u> <u>differential equations</u>:

$$\underline{x}'(t) = \underline{F}(t, \underline{x}(t))$$
$$\underline{x}(t_0) = \underline{x}_0$$

• We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

<u>Why we expect IVP's for first order systems of DE's to have unique solutions</u> $\underline{x}(t)$:

• From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{x}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{x}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:
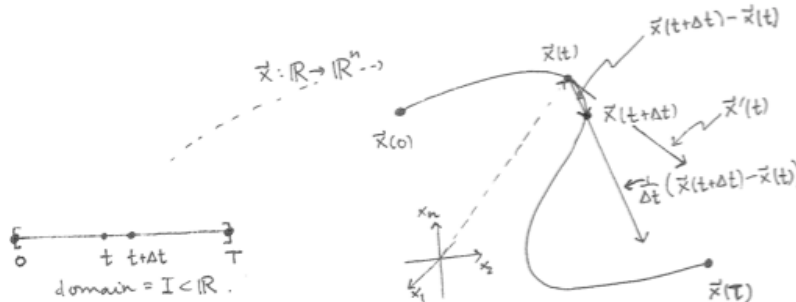
Analytically, the reason that the vector of derivatives $\underline{x}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{x}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t + \Delta t) \\ x_2(t + \Delta t) \\ \vdots \\ x_n(t + \Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t + \Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t + \Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t + \Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to <u>the IVP for a first order system</u> is that you know where you start ($\underline{x}(t_0) = \underline{x}_0$), and you know your

"velocity" vector (depending on time and current location) $\Rightarrow$ you <u>expect a unique solution!</u> (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! And this is what numerical solvers do.)

<u>Exercise 2)</u> Return to the page 1 tank example
$$x_1'(t) = -4 x_1 + 2 x_2$$
$$x_2'(t) = 4 x_1 - 2 x_2$$
$$x_1(0) = 9$$
$$x_2(0) = 0$$

<u>2a)</u> Interpret the parametric solution curve $\left[x_1(t), x_2(t) \right]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $\left[x_1(t), x_2(t) \right]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

<u>2b)</u> What are the apparent limiting solute amounts in each tank?

<u>2c)</u> How could your smart-alec younger sibling have told you the answer to <u>2b</u> without considering any differential equations or "velocity vector fields" at all?
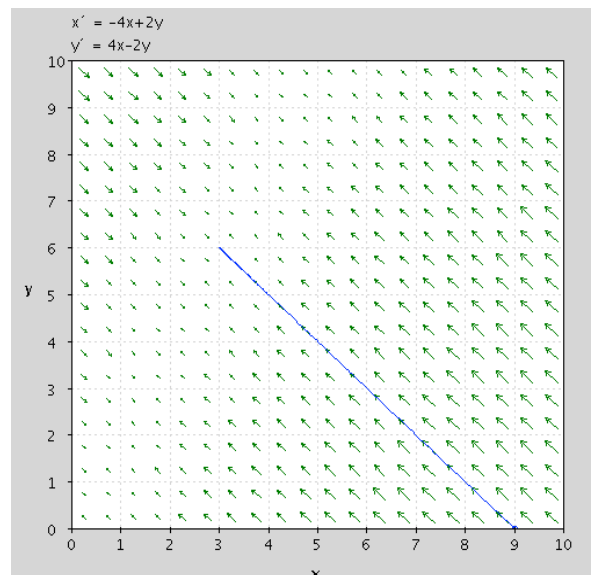
First order systems of differential equations of the form
$$\underline{x}'(t) = A\,\underline{x}$$
are called <u>linear homogeneous</u> systems of DE's. (Think of rewriting the system as
$$\underline{x}'(t) - A\,\underline{x} = \underline{0}$$
in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as
$$\underline{x}'(t) - A\,\underline{x} = \underline{f}(t)$$
or
$$\underline{x}'(t) = A\,\underline{x} + \underline{f}(t)$$

<u>Exercise 3a)</u> Show the space of solutions $\underline{x}(t)$ to the <u>homogeneous system</u> of DE's
$$\underline{x}'(t) = A\,\underline{x}$$
is a <u>subspace</u>, so linear combinations of solutions are solutions.
<u>3b)</u> Look for a basis of solutions of the form
$$\underline{x}(t) = e^{\lambda t}\underline{v},$$
where $\underline{v}$ is a constant vector. <u>Hint:</u> In order for such an $\underline{x}(t)$ to solve the DE it must be true that
$$\underline{x}'(t) = \lambda\,e^{\lambda t}\underline{v}$$
and
$$A\,\underline{x}(t) = A\,e^{\lambda t}\underline{v} = e^{\lambda t} A\,\underline{v}$$
must agree. *These functions of t will agree if and only* $\lambda\,\underline{v} = A\,\underline{v}$. So, it's time to find some eigenvalues and eigenvectors!

<u>3c)</u> Solve the initial value problem of <u>Exercise</u> 2!! Compare your solution $\underline{x}(t)$ to the parametric curve on the previous page.

<u>Exercise 4)</u> Lessons learned from tank example:
What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem
$$\underline{x}'(t) = A \underline{x}$$
$$\underline{x}(0) = \underline{x}_0 \in \mathbb{R}^n$$
using the method in <u>Exercise 3</u> ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)