Math 2250-4
Wed Nov 20
Announcements:
- Friday quiz covers 6.1-6.2.
- Laplace tranform portion of homework has been shortened
- Lab tomorrow focuses on Laplace transform.

- First, finish Tuesday's notes on eigenvalues, eigenvectors, eigenbases, and <u>diagonalizable matrices</u>. The square matrix $A_{n \times n}$ is <u>diagonalizable</u> if there is a basis for $\mathbb{R}^n$ (later in our course, for $\mathbb{C}^n$) consisting of eigenvectors of A.

<u>Exercise 5 (Tuesday November 19)</u> Check the examples below for diagonalizability:

9.18.1: $A := \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

9.18.2: $A := \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

9.18.3,4: $A := \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$, $|A - \lambda I| = (\lambda - 1)(\lambda - 4)$

$E_{\lambda = 1} = span\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $E_{\lambda = 4} = span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

9.19.1 $B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$, $|B - \lambda I| = -(\lambda - 2)^2(\lambda - 3)$

$E_{\lambda = 2} = span\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $E_{\lambda = 3} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

9.19.2: $A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

9.19.4: $C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $|C - \lambda I| = -(\lambda - 2)^2(\lambda - 3)$.

$E_{\lambda = 2} = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $E_{\lambda = 3} = span\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

• Now, we'll begin some post-exam Laplace transform material, useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle. We will continue this discussion on Friday, with a few more table entries including "the delta (impulse) function". On Monday we'll begin Chapter 7, systems of differential equations, and use the eigenvalue-eigenvector material from this week extensively.
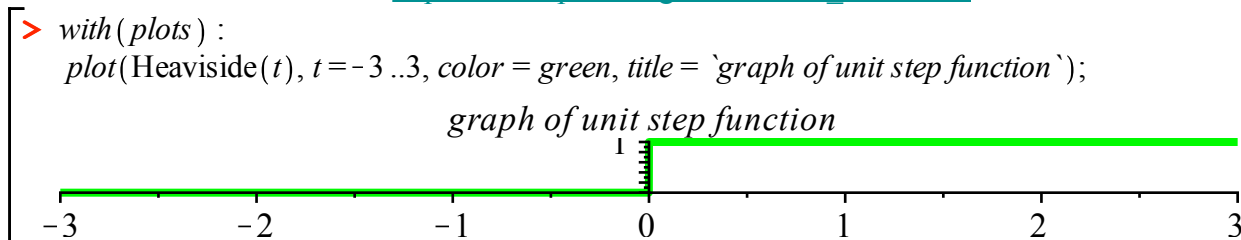
$f(t)$ with $\|f(t)\| \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\dfrac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$	$F(s)G(s)$	"convolution" for inverting products of Laplace transforms

The unit step function with jump at $t = 0$ is defined to be
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$
Its graph is shown below. Notice that this function is called the "Heaviside" function in Maple, after the person who popularized it (among a lot of other accomplishments) and not because it's heavy on one side.
http://en.wikipedia.org/wiki/Oliver_Heaviside

> $with\,(plots)$:
  $plot(\text{Heaviside}(t),\, t = -3\,..3,\, color = green,\, title = $ `graph of unit step function` $)$;
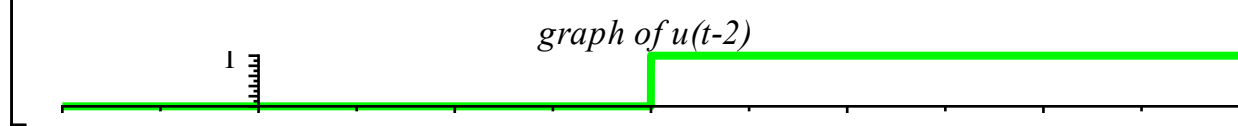

graph of unit step function

Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0; \text{ i.e. } t < a \\ 1, & t-a \geq 0; \text{ i.e. } t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:

> $plot(\text{Heaviside}(t-2), t=-1..5, color = green, title = \text{`graph of u(t-2)`});$

graph of u(t-2)

Exercise 1) Verify the table entries

$u(t-a)$ unit step function	$\dfrac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-as}F(s)$	more complicated on/off

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from last week - although you were asked to find the Laplace transform of that step function directly from the definition. In this week's homework assignment you will re-do that problem using unit step functions.
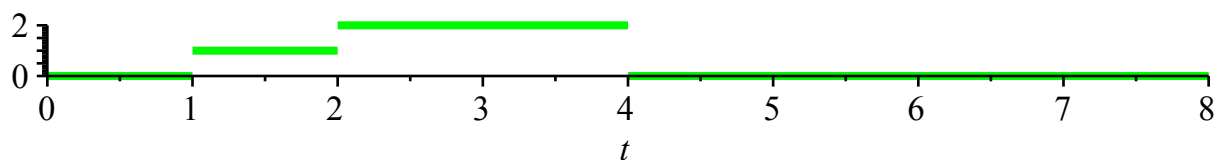
Setup: an under-employed   mathematician/engineer/scientist
                                  (your choice)

   likes to take his/her child to the swings...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$

$$\cdots \rightarrow x'' + g\frac{x}{L} = 0$$

$$\cdots \rightarrow mx'' + \frac{mg}{L}x = F_0 \cos\omega t \quad \leftarrow \text{parent forcing } (?!)$$

$x(t) = L \sin\theta(t)$
$\quad \cong L\theta \quad$ for small θ

so $\quad x'' \cong L\theta''$

$$\cdots \rightarrow x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

parent pushes sinusoidally for
exactly 5 cycles, and
with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
So $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds
⌣

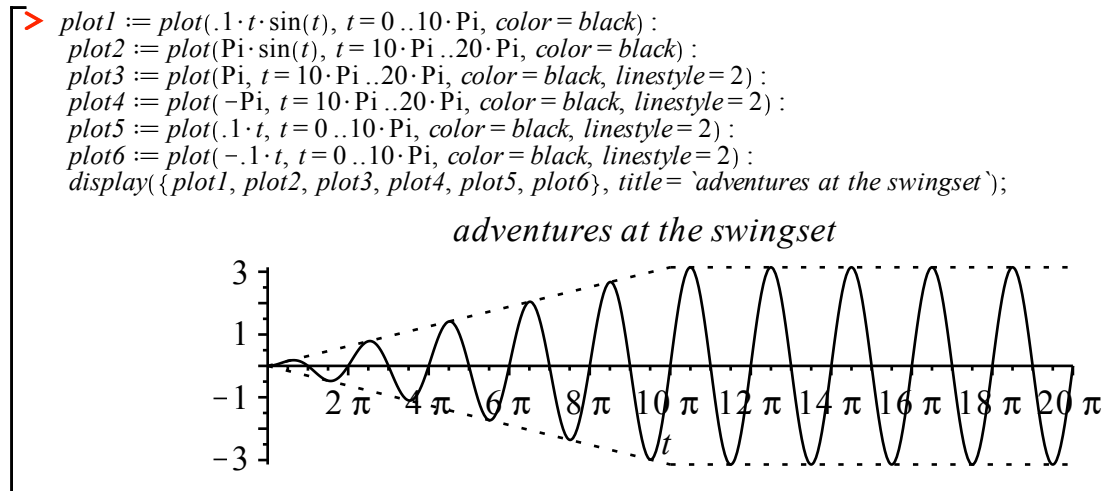Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)(1 - u(t - 10\pi))$$
$$x(0) = 0$$
$$x'(0) = 0$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```


adventures at the swingset

Alternate approach via Chapter 5, is doable, but involves more work:

step 1) solve

$$x''(t) + x(t) = .2 \cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \leq t \leq 10\,\pi$.

step 2) Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\,\pi)$$
$$y'(0) = x'(10\,\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

Laplace transform convolution

For Laplace transforms the convolution of f and g is defined to be

$$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Exercise 4) Show $f*g(t) = g*f(t)$ by doing a change of variables in the convolution integral.

$\int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

Exercise 5) Verify that the convolution integral table entry is correct, for

$$f(t) = \sin(t) \quad F(s) = \frac{1}{s^2+1}$$

$$g(t) = \cos(t) \quad G(s) = \frac{s}{s^2+1}$$

$$f*g(t) \qquad F(s)G(s) = \frac{s}{\left(s^2+1\right)^2}.$$

Hint: you might use the trig identity $\sin(\tau)^2 = \dfrac{1-\cos(2\,\tau)}{2}$.

```
> ∫₀ᵗ sin(τ)·cos(t−τ) dτ;

>
```

Why convolutions are so important in applications: (to be continued)...

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2\,X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} := F(s)W(s).$$

$\mathcal{L}^{-1}\{W(s)\}(t) = w(t)$ is called the weight function for the physical system, and because of the convolution table entry the solution is given by

$$x(t) = f*w(t) = w*f(t) = \int_0^t w(\tau)f(t-\tau)\,d\tau.$$

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system. Or, they may have a certain characteristics of the response function $x(t)$ that they would like to get, no matter the forcing function, and they design the weight function to get it.

$f(t)$, with $|f(t)| \leq Ce^{Mt}$	$F(s) := \int_0^{\infty} f(t)e^{-st}\, dt$ for $s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\dfrac{1}{s}$ $(s > 0)$	☐
t	$\dfrac{1}{s^2}$	☐
t^2	$\dfrac{2}{s^3}$	☐
t^n, $n \in \mathbb{N}$	$\dfrac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\dfrac{1}{s - \alpha}$ $(s > \Re(a))$	☐
$\cos(k\,t)$	$\dfrac{s}{s^2 + k^2}$ $(s > 0)$	☐
$\sin(k\,t)$	$\dfrac{k}{s^2 + k^2}$ $(s > 0)$	☐
$\cosh(k\,t)$	$\dfrac{s}{s^2 - k^2}$ $(s > k)$	☐
$\sinh(k\,t)$		☐
$e^{a\,t}\cos(k\,t)$	$\dfrac{k}{s^2 - k^2}$ $(s > k)$	☐
$e^{a\,t}\sin(k\,t)$	$\dfrac{(s - a)}{(s - a)^2 + k^2}$ $(s > a)$	☐
$e^{a\,t} f(t)$	$\dfrac{k}{(s - a)^2 + k^2}$ $(s > a)$	☐
	$F(s - a)$	
$u(t - a)$	$\dfrac{e^{-as}}{s}$	
$f(t - a)\, u(t - a)$	$e^{-as} F(s)$	
$\delta(t - a)$	e^{-as}	
$f'(t)$	$s\, F(s) - f(0)$	☐
$f''(t)$	$s^2 F(s) - s\, f(0) - f'(0)$	☐
$f^{(n)}(t)$, $n \in \mathbb{N}$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$	☐
$\displaystyle\int_0^t f(\tau)\, d\tau$	$\dfrac{F(s)}{s}$	☐
$t\, f(t)$	$-F'(s)$	☐
$t^2 f(t)$	$F''(s)$	☐
$t^n f(t)$, $n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$	☐
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^{\infty} F(\sigma)\, d\sigma$	

$t \cos(k\,t)$	$\dfrac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$	☐
$\dfrac{1}{2\,k} t \sin(k\,t)$	$\dfrac{s}{\left(s^2 + k^2\right)^2}$	☐
$\dfrac{1}{2\,k^3}\left(\sin(k\,t) - k\,t \cos(k\,t)\right)$	$\dfrac{1}{\left(s^2 + k^2\right)^2}$	☐
$t\,e^{a\,t}$	$\dfrac{1}{(s-a)^2}$	☐
$t^n\,e^{a\,t},\; n \in \mathbb{Z}$	$\dfrac{n!}{(s-a)^{n+1}}$	☐
$\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\dfrac{1}{1 - e^{-ps}}\displaystyle\int_0^p f(t)e^{-s\,t}\,dt$	

Laplace transform table