Math 2250-4
Tues Nov 19
6.1-6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability of matrices.

Recall from yesterday,

<u>Definition</u>: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for some scalar λ and vector $\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$ then $\underline{\boldsymbol{v}}$ is called an <u>eigenvector</u> of $\underline{A}$, and λ is called the <u>eigenvalue</u> of $\underline{\boldsymbol{v}}$ (and an eigenvalue of A).

- For general matrices, the eigenvector equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ can be rewritten as

$$(A - \lambda\, I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}} \ .$$

The only way such an equation can hold for $\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$ is if the matrix $(A - \lambda\, I)$ does not reduce to the identity matrix. In other words - $det(A - \lambda\, I)$ must equal zero. Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the <u>characteristic polynomial</u>

$$p(\lambda) = det(A - \lambda\, I) \ .$$

- So, the first step in finding eigenvectors for A is actually to find the eigenvalues - by finding the characteristic polynomial and its roots λ_j .

- For each root λ_j the matrix $A - \lambda_j\, I$ will not reduce to the identity, and the solution space to

$$(A - \lambda_j\, I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$$

will be at least one-dimensional, and have a basis of one or more eigenvectors. Find such a basis for this λ_j <u>eigenspace</u> $E_{\lambda = \lambda_j}$ by reducing the homogeneous matrix equation

$$(A - \lambda_j\, I)\underline{\boldsymbol{v}} = \underline{\mathbf{0}} \ ,$$

backsolving, and extracting a basis.

<u>Exercise 1a)</u> Do this exercise from yesterday's notes. The matrix for which we want to find eigenvalues and eigenvectors (eigenspace bases) is

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} .$$

<u>step 1)</u>

$$|B - \lambda\, I| = \begin{bmatrix} 4 - \lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3 - \lambda \end{bmatrix} =$$

step 2) Find bases for the eigenspaces. Talk about how you are able to use the fact that homogeneous solutions to matrix equations correspond to column dependencies, in order to read off eigenspace bases more quickly than by backsolving.... ask questions if this is not clear, because you'll be computing a lot of eigenvalues and eigenvectors in the next chapters and you'll save a lot of time if you get comfortable with this shortcut. (Of course, for most larger matrices one just uses technology for eigenvalue/eigenvector computation.)

1b) Notice that you can construct a basis for $\mathbb{R}^3$ by combining your eigenspace bases from 1a. Use this fact to describe the geometry of the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$T(\underline{\boldsymbol{x}}) = B\,\underline{\boldsymbol{x}}\,.$$

Your algebraic work in 1a is related to the output below:

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> with(LinearAlgebra) :
  B := Matrix(3, 3, [4, −2, 1, 2, 0, 1, 2, −2, 3]);
  Eigenvectors(B);
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$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \textbf{(1)}$$

Exercise 2) If your matrix A is diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from yesterday, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a 3×3 diagonal matrix, so that in the future you can just read of the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

step 1) Find the roots of the characteristic polynomial $det(A - \lambda I)$.
step 2) Find the eigenspace bases, assuming the values of a_{11}, a_{22}, a_{33} are distinct (all different). What if $a_{11} = a_{22}$ but these values do not equal a_{33} ?

In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 1, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P). Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

Exercise 3) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

Exercise 4) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable.

<u>Facts about diagonalizability</u> (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n \left(\lambda - \lambda_1\right)^{k_1} \left(\lambda - \lambda_2\right)^{k_2} \ldots \left(\lambda - \lambda_m\right)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda = \lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called <u>defective.</u>
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ each eigenspace is forced to be

1-dimensional since $k_1 + k_2 + \ldots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

<u>Exercise 5)</u> How do the examples from today and yesterday compare with the general facts about diagonalizability?