Math 2250-4
Mon Nov 18

We will still do more Laplace transform applications - in fact that will be the focus of the lab this Thursday. But the homework due this Friday will mostly be from Chapter 6, so we will get a good start on that material today, before returning to Laplace transform material towards the end of Tuesday's lecture.

6.1-6.2 Eigenvalues and eigenvectors for square matrices.

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear systems of differential equations, in Chapter 7. Such systems of DE's arise naturally in the contexts of

- coupled input-output models, with several components.
- coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

Example Consider the matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

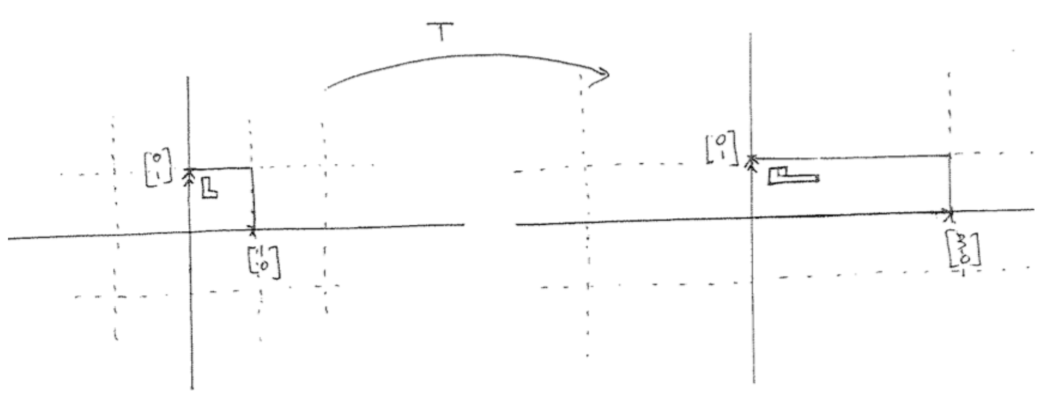
Notice that for the standard basis vectors $\underline{e}_1 = [1, 0]^T$, $\underline{e}_2 = [0\ , 1]^T$

$$T(\underline{e}_1) = 3\underline{e}_1$$
$$T(\underline{e}_2) = \underline{e}_2\ .$$

The facts that T is linear and that it transforms $\underline{e}_1$, $\underline{e}_2$ by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(x_1\underline{e}_1 + x_2\underline{e}_2) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2)$$
$$= x_1(3\underline{e}_1) + x_2(1\underline{e}_2)\ .$$

In other words, T stretches by a factor of 3 in the $\underline{e}_1$ direction, and by a factor of 1 in the $\underline{e}_2$ direction, transforming a square grid in the domain into a parallel rectangular grid in the image:



Exercise 1) Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 2) And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for a scalar λ and a vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$. (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

• In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

Exercise 3) Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors $\underline{\boldsymbol{x}}$ and computing $A\,\underline{\boldsymbol{x}}$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

How to find eigenvalues and eigenvectors (including eigenspaces) systematically:

If

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\,I\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\,I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}\,.$$

As we know, this last equation can have non-zero solutions $\underline{\boldsymbol{v}}$ if and only if the matrix $(A - \lambda\,I)$ is not invertible, i.e.

$$\Leftrightarrow det(A - \lambda\,I) = 0\,.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\,I)\,.$$

If $A_{n \times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the characteristic polynomial of the matrix A.

- λ_j can be an eigenvalue for some non-zero eigenvector $\underline{\boldsymbol{v}}$ if and only if it's a root of the characteristic polynomial, i.e. $p(\lambda_j) = 0$. For each such root, the homogeneous solution space of vectors $\underline{\boldsymbol{v}}$ solving

$$(A - \lambda_j\,I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

will be eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\,I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue λ_j is called the λ_j eigenspace, and denoted by E_{λ_j}. (We include the zero vector in E_{λ_j}.) The basis of eigenvectors is called an eigenbasis for E_{λ_j}.

Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues.
(ii) for each eigenvalue, find bases for the corresponding eigenspaces.
(iii) Can you describe the transformation $T(\underline{\mathbf{x}}) = B\underline{\mathbf{x}}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

Your solution will be related to the output below:

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> with(LinearAlgebra) :
  B := Matrix(3, 3, [4,-2, 1, 2, 0, 1, 2,-2, 3]);
  Eigenvectors(B);
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$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

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In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the <u>geometry</u> of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is <u>diagonalizable.</u> Here's an example of a matrix which is NOT diagonalizable:

<u>Exercise 7</u>: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.