Math 2250-4
Wed Nov 13

HW due tomorrow Thursday at start of labs! Review in labs on Thursday! Exam on Friday in class, 8:30 -9:30 a.m.! Go over last spring exam 2 and answer hw questions, Wed 4:30-6:00 Room WEB L103.

- Today we'll finish Tuesday's notes and make sure we understand how to set up and solve partial fraction problems, since these are often key to using Laplace transform tables in finding inverse Laplace transforms.

- Then, we'll begin some post-exam Laplace transform material, useful in systems where we turn forcing functions on and off:

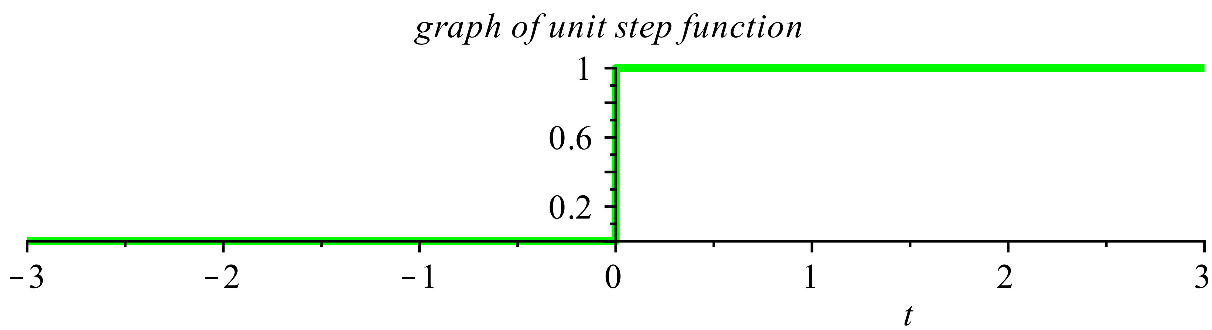
$f(t)$ with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off

The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

Its graph is shown below. Notice that this function is called the "Heaviside" function in Maple, after the person who popularized it (among a lot of other accomplishments) and not because it's heavy on one side.
http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots) :
  plot(Heaviside(t), t=-3 ..3, color = green, title = `graph of unit step function`);
```



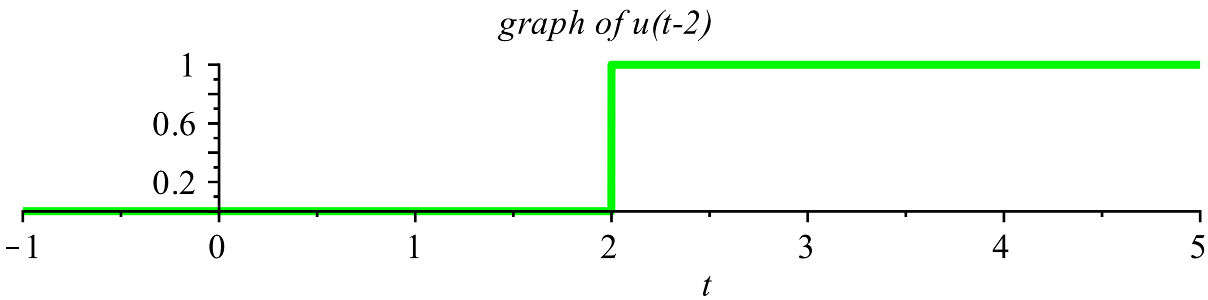
Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0; \text{ i.e. } t < a \\ 1, & t-a \geq 0; \text{ i.e. } t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:

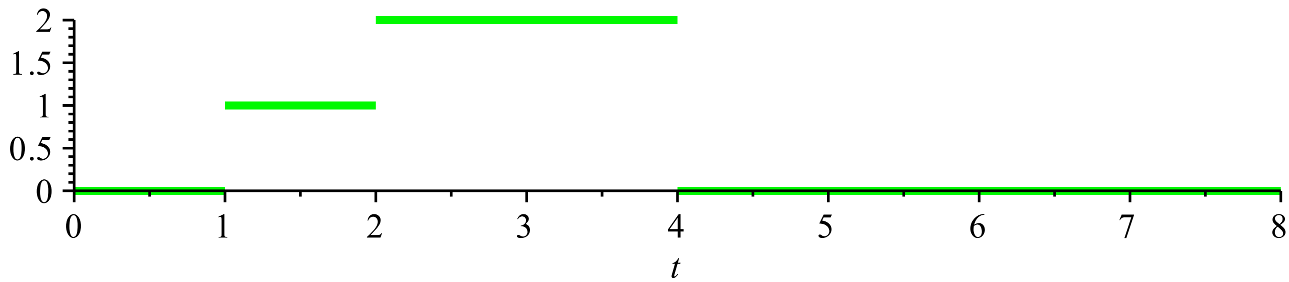
```
> plot(Heaviside(t - 2), t=-1..5, color = green, title = `graph of u(t-2)`);
```



Exercise 1) Verify the table entries

$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)



> $plot(\text{Heaviside}(t-1) + \text{Heaviside}(t-2) - 2\cdot\text{Heaviside}(t-4), t = 0..8, color = green);$
> $with(inttrans):$
> $laplace(\text{Heaviside}(t-1) + \text{Heaviside}(t-2) - 2\cdot\text{Heaviside}(t-4), t, s);$

Setup: an under-employed mathematician/engineer/scientist (your choice)

likes to take his/her child to the swings...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$$L\theta'' + g\theta = 0$$

$$\dashrightarrow \; x'' + g\frac{x}{L} = 0$$

$$\dashrightarrow \; mx'' + \frac{mg}{L}x = F_0 \cos \omega t \quad \leftarrow \text{parent forcing (?!)}$$

$x(t) = L \sin\theta(t)$
$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$$\dashrightarrow \; x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

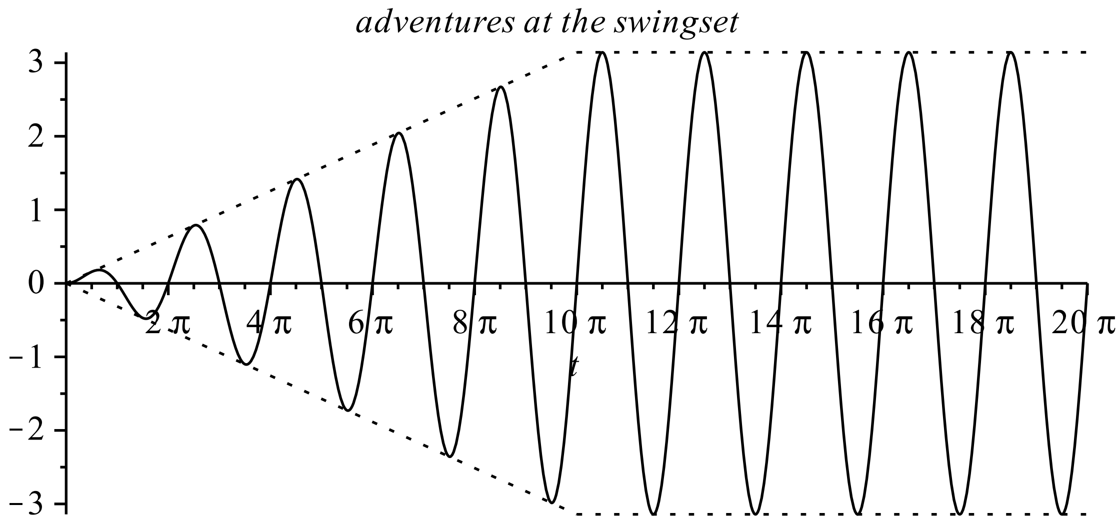
Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\left(1 - u(t - 10\pi)\right)$$
$$x(0) = 0$$
$$x'(0) = 0$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 5:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\,\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\,\pi)$$
$$y'(0) = x'(10\,\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

$f(t)$, with $\lvert f(t)\rvert \le Ce^{M t}$	$F(s) := \int_0^\infty f(t)e^{-s t}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
t^n, $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s - \alpha}$ $(s > \Re(a))$	☐
$\cos(k t)$	$\frac{s}{s^2 + k^2}$ $(s > 0)$	☐
$\sin(k t)$	$\frac{k}{s^2 + k^2}$ $(s > 0)$	☐
$\cosh(k t)$	$\frac{s}{s^2 - k^2}$ $(s > k)$	☐
$\sinh(k t)$	$\frac{k}{s^2 - k^2}$ $(s > k)$	☐
$e^{a t}\cos(k t)$	$\frac{(s-a)}{(s-a)^2 + k^2}$ $(s > a)$	☐
$e^{a t}\sin(k t)$	$\frac{k}{(s-a)^2 + k^2}$ $(s > a)$	☐
$e^{a t} f(t)$	$F(s - a)$	☐
$u(t - a)$	$\frac{e^{-a s}}{s}$	
$f(t - a)\, u(t - a)$	$e^{-a s}F(s)$	
$\delta(t - a)$	$e^{-a s}$	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t)$, $n \in \mathbb{N}$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐
$\int_0^t f(\tau)\,d\tau$		☐

	$\dfrac{F(s)}{s}$	
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\dfrac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\dfrac{1}{2\,k}\,t\sin(k\,t)$ $\dfrac{1}{2\,k^3}\left(\sin(k\,t) - k\,t\cos(k\,t)\right)$ $t\,e^{a\,t}$ $t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$ $\dfrac{s}{(s^2 + k^2)^2}$ $\dfrac{1}{(s^2 + k^2)^2}$ $\dfrac{1}{(s - a)^2}$ $\dfrac{n!}{(s - a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\dfrac{1}{1 - e^{-ps}}\int_0^p f(t)e^{-s\,t}\,dt$	

Laplace transform table