

HW due Thursday at start of labs! Review in labs on Thursday! Exam on Friday in class, 8:30-9:30 a.m.!  
Go over last spring exam 2 and answer hw questions, Wed 4:30-6:00 Room TBA.

10.2-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.

Today we'll continue to fill in the Laplace transform table, and to use the table entries to solve linear differential equations. One focus today will be to review partial fractions, since the table entries are set up precisely to show the inverse Laplace transforms of the components of partial fraction decompositions.

Exercise 1) Check why this table entry is true - notice that it generalizes how the Laplace transforms of  $\cos(kt)$ ,  $\sin(kt)$  are related to those of  $e^{at}\cos(kt)$ ,  $e^{at}\sin(kt)$ :

$e^{at}f(t)$	$F(s-a)$
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Exercise 2) Verify the table entry

$t^n, n \in \mathbb{Z}$	$\frac{n!}{s^{n+1}}$
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by applying one of the results from yesterday:

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{(n-1)}(0).$$

Exercise 3) Combine 1,2, to get

$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
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A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$tf(t)$	$-F'(s)$
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Here's how we get it:

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st} dt \\ \Rightarrow \frac{d}{ds}F(s) &= \frac{d}{ds} \int_0^\infty f(t)e^{-st} dt = \int_0^\infty \frac{d}{ds}f(t)e^{-st} dt. \end{aligned}$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-st} dt = \int_0^\infty f(t)(-t)e^{-st} dt = -\mathcal{L}\{tf(t)\}(s) \quad \square.$$

For resonance and other applications ...

Exercise 4) Use  $\mathcal{L}\{t f(t)\}(s) = -F'(s)$  or Euler's formula and  $\mathcal{L}\{t e^{at}\}(s) = \frac{1}{(s-a)^2}$  to show

a)  $\mathcal{L}\{t \cos(kt)\}(s) = \frac{s^2 - k^2}{(s^2 + k^2)^2}$

b)  $\mathcal{L}\left\{\frac{1}{2k} t \sin(kt)\right\}(s) = \frac{s}{(s^2 + k^2)^2}$

c) Then use a and the identity

$$\frac{1}{(s^2 + k^2)^2} = \frac{1}{2k^2} \left( \frac{s^2 + k^2}{(s^2 + k^2)^2} - \frac{s^2 - k^2}{(s^2 + k^2)^2} \right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\}(t) = \frac{1}{2k^2} \left( \frac{1}{k} \sin(kt) - t \cos(kt) \right).$$

Notice how the Laplace transform table is set up to use partial fraction decompositions. And be amazed at how it lets you quickly deduce the solutions to important DE IVPs, like this resonance problem:

Exercise 5a) Use Laplace transforms to write down the solution to

$$\begin{aligned}x''(t) + \omega_0^2 x(t) &= F_0 \sin(\omega_0 t) \\x(0) &= x_0 \\x'(0) &= v_0.\end{aligned}$$

what phenomenon do the solutions to this IVP exhibit?

5b) Use Laplace transforms to solve the general undamped forced oscillation problem, when  $\omega \neq \omega_0$  :

$$\begin{aligned}x''(t) + \omega_0^2 x(t) &= F_0 \sin(\omega t) \\x(0) &= x_0 \\x'(0) &= v_0\end{aligned}$$

$f(t), \text{ with }  f(t)  \leq C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t} dt \text{ for } s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	<input type="checkbox"/>
$1$ $t$ $t^2$ $t^n, n \in \mathbb{N}$	$\frac{1}{s} \quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	<input type="checkbox"/>
$e^{\alpha t}$	$\frac{1}{s - \alpha} \quad (s > \Re(a))$	<input type="checkbox"/>
$\cos(k t)$ $\sin(k t)$ $\cosh(k t)$ $\sinh(k t)$ $e^{a t} \cos(k t)$ $e^{a t} \sin(k t)$ $e^{a t} f(t)$	$\frac{s}{s^2 + k^2} \quad (s > 0)$ $\frac{k}{s^2 + k^2} \quad (s > 0)$ $\frac{s}{s^2 - k^2} \quad (s > k)$ $\frac{k}{s^2 - k^2} \quad (s > k)$ $\frac{(s - a)}{(s - a)^2 + k^2} \quad (s > a)$ $\frac{k}{(s - a)^2 + k^2} \quad (s > a)$ $F(s - a)$	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
$f'(t)$ $f''(t)$ $f^{(n)}(t), n \in \mathbb{N}$ $\int_0^t f(\tau) d\tau$	$s F(s) - f(0)$ $s^2 F(s) - s f(0) - f'(0)$ $s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
$t f(t)$ $t^2 f(t)$ $t^n f(t), n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$\frac{-F'(s)}{F''(s)}$ $(-1)^n F^{(n)}(s)$ $\int_s^\infty F(\sigma) d\sigma$	
$t \cos(k t)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$	

$\frac{1}{2k} t \sin(kt)$ $\frac{1}{2k^3} (\sin(kt) - kt \cos(kt))$ $t e^{at}$ $t^n e^{at}, n \in \mathbb{Z}$	$\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s - a)^2}$ $\frac{n!}{(s - a)^{n+1}}$	
more after the midterm!		

### Laplace transform table

Exercise 5) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$\begin{aligned}
 x''(t) + 4x(t) &= 8te^{2t} \\
 x(0) &= 0 \\
 x'(0) &= 1
 \end{aligned}$$

Exercise 6a) What is the form of the partial fractions decomposition for

$$X(s) = \frac{-356 + 45s - 100s^2 - 4s^5 - 9s^4 + 39s^3 + s^6}{(s-3)^3((s+1)^2+4)(s^2+4)}.$$

6b) Have Maple compute the precise partial fractions decomposition.

6c) What is  $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$  ?

6d) Have Maple compute the inverse Laplace transform directly.

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[> X := s -> (-356 + 45*s - 100*s^2 - 4*s^5 - 9*s^4 + 39*s^3 + s^6) /
    (s - 3)^3 * ((s + 1)^2 + 4) * (s^2 + 4)^2;
[> convert(X(s), parfrac, s);
[> with(inttrans);
[> invlaplace(X(s), s, t);
[>
```