Math 2250-4
Mon Nov 11

Homework office hours Tuesday TBA, for HW due Wednesday.
Go over last spring exam 2, Wednesday or Thursday 4:30-6:00 Room TBA.

10.1-10.3 Laplace transform, and application to DE IVPs, especially those in Chapter 5. Today we'll continue to fill in the Laplace transform table (on page 2), and to use the table entries to solve linear differential equations.

Exercise 1) (to review) Use the table to compute

1a) $\mathcal{L}\{4 - 5\cos(3t) + 2e^{-4t}\sin(12t)\}(s)$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{1}{s^2 + 2s + 5}\right\}(t)$.

Exercise 2a) (to review) Use Laplace transforms to solve the IVP we didn't get to on Friday, for an underdamped, unforced oscillator DE. Compare to Chapter 5 method.

$$x''(t) + 6x'(t) + 34x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	
t^2	$\frac{2}{s^3}$	
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(\alpha))$	☐
$\cos(k t)$	$\frac{s}{s^2 + k^2}$ $(s > 0)$	☐
$\sin(k t)$	$\frac{k}{s^2 + k^2}$ $(s > 0)$	☐
$\cosh(k t)$	$\frac{s}{s^2 - k^2}$ $(s > k)$	
$\sinh(k t)$	$\frac{k}{s^2 - k^2}$ $(s > k)$	
$e^{a t}\cos(k t)$	$\frac{(s-a)}{(s-a)^2 + k^2}$ $(s > a)$	☐
$e^{a t}\sin(k t)$	$\frac{k}{(s-a)^2 + k^2}$ $(s > a)$	☐
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$	
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	
$t\, f(t)$	$-F'(s)$	
$t^2 f(t)$	$F''(s)$	
$t^n f(t),\ n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$	
$t\cos(k t)$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$	
$\frac{1}{2k} t \sin(k t)$	$\frac{s}{(s^2 + k^2)^2}$	

$\frac{1}{2k^3}(\sin(k t) - k t \cos(k t))$	$\frac{1}{(s^2 + k^2)^2}$	
$e^{a t} f(t)$ $t e^{a t}$ $t^n e^{a t}, n \in \mathbb{Z}$	$F(s - a)$ $\frac{1}{(s - a)^2}$ $\frac{n!}{(s - a)^{n+1}}$	
more after exam!		

Laplace transform table

work down the table ...

3a) $\mathcal{L}\{cosh(k t)\}(s) = \frac{s}{s^2 - k^2}$

3b) $\mathcal{L}\{\sinh(k t)\}(s) = \frac{k}{s^2 - k^2}$.

Exercise 4) Recall we used integration by parts on Friday to derive

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0) .$$

Use that identity to show

a) $\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s f(0) - f'(0)$,

b) $\mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$,

c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0), n \in \mathbb{N}$.

d) $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \frac{F(s)}{s}$.

These are the identities that make Laplace transform work so well for initial value problems such as we studied in Chapter 5.

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 4d.

b) using partial fractions.

Exercise 6) Show $\mathcal{L}\{t^n\}(s)=\frac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4.

Tomorrow we will finish filling in the Laplace transform table entries. Notice how the Laplace transform table is set up to use partial fraction decompositions for $X(s)$ and linearity in order to find the inverse Laplace transforms $x(t)$. And be amazed at how the table lets you quickly deduce the solutions to important DE IVPs, like we studied in Chapter 5.

Exercise 7)

a) Use Laplace transforms to write down the solution to

$$\begin{aligned} x''(t) + \omega_0^2 x(t) &= F_0 \sin(\omega_0 t) \\ x(0) &= x_0 \\ x'(0) &= v_0 . \end{aligned}$$

what phenomena do solutions to this DE illustrate (even though we're forcing with $\sin(\omega_0 t)$ rather than $\cos(\omega_0 t)$)?

b) Use Laplace transforms to derive the IVP solution below, in the case $\omega \neq \omega_0$.

$$\begin{aligned} x''(t) + \omega_0^2 x(t) &= F_0 \sin(\omega t) \\ x(0) &= x_0 \\ x'(0) &= v_0 . \end{aligned}$$