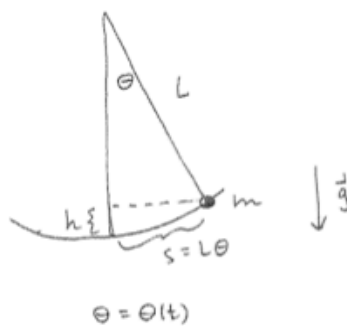


Math 2250-4  
 Fri Nov 1  
 Experiment day :-)  
 Superquiz day :-/

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods  $T = \frac{2\pi}{\omega_0}$  correspond to experimental reality.

① pendulum



conservative system  $KE + PE = \text{const.}$

$$\frac{1}{2}mv^2 + mgh = \text{const}$$

$$s = L\theta$$

$$v = \frac{ds}{dt} = L\theta'(t)$$

$$h = L - L\cos\theta = L(1 - \cos\theta)$$

$$\text{so, } \frac{1}{2}mL^2(\theta'(t))^2 + mgL(1 - \cos(\theta(t))) \equiv \text{const}$$

$$D_t: mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$$

$$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}} (L\theta'' + g\sin\theta) \equiv 0$$

$\neq 0$  except  
at isolated  
times

$\sim$  deduce eqn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

(linearize)

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

$\hookrightarrow$  non-linear DE

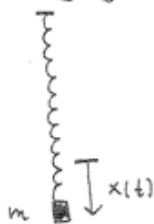
$$\text{but } \sin\theta = \theta - \frac{\theta^3}{3!} + \dots$$

$$\sin\theta \approx \theta \quad \theta \text{ small}$$

is excellent approx

(alternating series test)

② hanging mass-spring:



$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Why don't you see gravity  $g$   
in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
[> restart :
  Digits := 4 :

> L := 1.526;
  g := 9.806;
  ω := √(g/L); # radians per second
  f := evalf(ω/(2·Pi)); # cycles per second
  T := 1/f; # seconds per cycle
```

Experiment:

Mass-spring:

compute Hooke's constant:

```
[> 98.7 - 83.4; #displacement from extra 50g
> k := .05·9.806 / .153; # solve k·x=m·g for k.
> m := .1; # mass for experiment is 100g
  ω := √(k/m); # predicted angular frequency
  f := evalf(ω/(2·Pi)); # predicted frequency
  T := 1/f; # predicted period
```

Experiment:

We neglected the  $KE_{spring}$ , which is small but could be adding inertia to the system and slowing down the oscillations. We can account for this:

### Improved mass-spring model

Normalize  $TE = KE + PE = 0$  for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k s \, ds = \frac{1}{2} k x^2, \quad KE_{mass} = \frac{1}{2} m (x'(t))^2, \quad KE_{spring} = ???$$

How to model  $KE_{spring}$  ? Spring is at rest at top (where it's attached to bar), moving with velocity  $x'(t)$  at bottom (where it's attached to mass). Assume it's moving with velocity  $\mu x'(t)$  at location which is fraction  $\mu$  of the way from the top to the mass. Then we can compute  $KE_{spring}$  as an integral with respect to  $\mu$ , as the fraction varies  $0 \leq \mu \leq 1$  :

$$\begin{aligned} KE_{spring} &= \int_0^1 \frac{1}{2} (\mu x'(t))^2 (m_{spring} d\mu) \\ &= \frac{1}{2} m_{spring} (x'(t))^2 \int_0^1 \mu^2 d\mu = \frac{1}{6} m_{spring} (x'(t))^2 . \end{aligned}$$

Thus

$$TE = \frac{1}{2} \left( m + \frac{1}{3} m_{spring} \right) (x'(t))^2 + \frac{1}{2} k x^2 = \frac{1}{2} M (x'(t))^2 + \frac{1}{2} k x^2 ,$$

where

$$M = m + \frac{1}{3} m_{spring}$$

$$D_t(TE) = 0 \Rightarrow$$

$$\begin{aligned} M x'(t) x''(t) + k x(t) x'(t) &= 0 . \\ x'(t) (M x'' + k x) &= 0 . \end{aligned}$$

Since  $x'(t) = 0$  only at isolated  $t$ -values, we deduce that the corrected equation of motion is

$$(M x'' + k x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3} m_{spring}}} .$$

Does this lead to a better comparison between model and experiment?

```
[> ms := .011; # spring has mass 11g
  M := m + 1/3 * ms; # "effective mass"
]
[> omega := sqrt(k/M); # predicted angular frequency
  f := evalf(omega/(2*Pi)); # predicted frequency
  T := 1/f; # predicted period
]>
```