

Math 2250-4
Week 9 concepts and homework, due November 1.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems. (Well, this Friday we'll be having a super quiz, covering 4.1-4.4, 5.1-5.3, and maybe part of 5.4.)

5.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

5.3: 3, 9, 11, 23, 27.

w9.1) Do the following problems for homogeneous linear differential equations by hand. They are testing your ability to use the algorithm for finding bases for the solutions spaces, based on the characteristic polynomial. Check your work with Maple (or other software). In Maple you will want to use the "dsolve" command to check differential equation solutions, and may want to use the "factor" command to check your factorizations of the characteristic polynomials. Hand in a printout of your computer verifications for the differential equation solutions, along with your written work.

a) Find the general solution to the differential equation for $y(x)$

$$y^{(3)} - 4y'' + y' + 6y = 0.$$

Hint: Find a root r_1 of the cubic characteristic polynomial, then divide it by $(r - r_1)$ to get a quotient quadratic polynomial.

b) Find the general solution to the differential equation for $x(t)$

$$x'' + 6x' + 13x = 0.$$

Hint: completing the square works well here - probably better than the quadratic formula.

c) Solve the initial value problem for the differential equation in **b**, with $x(0) = 2$, $x'(0) = 0$.

d) Find the general solution to the differential equation for $y(x)$

$$y^{(4)} - 64y' = 0.$$

e) Find the general solution to the differential equation for $y(x)$

$$y^{(5)} - 8y^{(3)} + 16y' = 0.$$

w9.2) Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

is extremely useful in higher mathematics/science/engineering. In class we discuss how this definition is motivated by Taylor series. Amazingly, the rule of exponents is true for such expressions. In other words

$$e^{i\alpha + i\beta} = e^{i\alpha} e^{i\beta}.$$

Check this identity by rewriting the left side as $e^{i(\alpha + \beta)}$ and then using Euler's formula to expand both sides. You'll notice that the identity is true because of the addition angle formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (And so this gives a good way to recover the trig. identities if you happen to forget them.)

w9.3) We explore why the general algorithm for finding a basis of solutions for constant coefficient homogeneous linear differential equations yields independent functions, for an illustrative example.

a) What 4th order constant coefficient homogeneous linear DE has characteristic polynomial

$$p(r) = (r + 4)^2(r^2 + 9) ?$$

b) According to our general algorithm, write down a basis for the 4-dimensional solution space to the DE in part **a**.

c) Prove that the four functions you wrote down in part **b** actually are linearly independent (so automatically span, so a basis). Hint: Use a combination of the techniques we discussed on Friday October 2, rather than attempting a Wronskian computation. Notice that the solutions exist on the entire real line $-\infty < x < \infty$. (We will also have related discussions on Monday October 28.)

In the next exercise you will continue to explore the reasoning that leads to the "magic" general algorithm for solving constant coefficient homogeneous linear DEs in the case that the characteristic polynomial has repeated roots. These ideas are also discussed in section 5.3, and it may help you to read the book's discussion. Here's a summary: Begin with the derivative operator D defined by $D(y) := y'$. Then the second derivative operator is $D^2(y) := D \circ D(y)$, etc. So for $y = y(x)$ and

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y$$

we may write

$$L = D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0I$$

where I is the identity operator, $I(y) := y$. In other words, we are writing the operator L as a linear combination of various derivative operators. It turns out that any factorization of the characteristic polynomial $p(r)$ leads to a corresponding factorization of L . For example, consider the second order case with real roots,

$$L = D^2 + a_1D + a_0I.$$

If $p(r)$ factors with real roots,

$$p(r) = r^2 + a_1r + a_0 = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1r_2$$

Then L factors the same way, as a composition of first order operators:

$$L = D^2 - (r_1 + r_2)D + r_1r_2I = (D - r_1I) \circ (D - r_2I).$$

(To check this is to work from right to left above:

$$\begin{aligned} (D - r_1I) \circ (D - r_2I) &= D \circ (D - r_2I) - r_1I \circ (D - r_2I) \\ &= D \circ D - D \circ r_2I - r_1I \circ D - r_1I \circ (-r_2I) \\ &= D^2 - r_2D - r_1D - r_1r_2I = L. \end{aligned}$$

Now, compute

$$(D - r_1I)(e^{r_1x}) = D(e^{r_1x}) - r_1e^{r_1x} = r_1e^{r_1x} - r_1e^{r_1x} = 0.$$

Thus this factoring of L as

$$L = (D - r_1I) \circ (D - r_2I) = (D - r_2I) \circ (D - r_1I)$$

is (another reason) why e^{r_1x}, e^{r_2x} both satisfy $L(y) = 0$ in case $r_1 \neq r_2$ are roots of the characteristic polynomial, since each exponential is transformed into the zero function by one of the linear factors of L . The case $r_1 = r_2$ is the double root case,

$$L = (D - r_1 I) \circ (D - r_1 I) = (D - r_1 I)^2.$$

More generally, if the characteristic polynomial for the n^{th} order homogeneous DE factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \dots (r - r_m)^{k_m}$$

then the corresponding operator L factors as a composition

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \dots \circ (D - r_m I)^{k_m}.$$

w9.4)

a) Let $f(x)$ be any differentiable function. Check that

$$(D - r_1 I)(f(x)e^{r_1 x}) = f'(x)e^{r_1 x}.$$

b) Deduce that

$$\begin{aligned} (D - r_1 I)e^{r_1 x} &= 0 \\ (D - r_1 I)(xe^{r_1 x}) &= e^{r_1 x} \\ (D - r_1 I)x^2 e^{r_1 x} &= 2x e^{r_1 x} \\ (D - r_1 I)x^k e^{r_1 x} &= kx^{k-1} e^{r_1 x}. \end{aligned}$$

c) Use your work from **b** to explain why

$$\begin{aligned} (D - r_1 I)^2(xe^{r_1 x}) &= 0 \\ (D - r_1 I)^3(x^2 e^{r_1 x}) &= 0 \\ (D - r_1 I)^k(x^{k-1} e^{r_1 x}) &= 0, \quad k \in \mathbb{N}. \end{aligned}$$

d) Explain why if L has a factor

$$(D - r_1 I)^{k_1}$$

with $k_1 > 1$, then the k_1 functions

$$e^{r_1 x}, x e^{r_1 x}, \dots, x^{k_1 - 1} e^{r_1 x}$$

as we wrote down in the algorithm for finding a solution space basis, do all indeed satisfy $L(y) = 0$.

5.4: mechanical vibrations: setting up and solving differential equations and initial value problems for mechanical vibrations; underdamped, critically damped, and overdamped vibration problems; applications to pendulums, mass-spring configurations, and related problems; understanding and using phase-amplitude form for sinusoidal functions.

5.4: 3, **4**, 5, **6**, **10**, (in **10** the weight of the buoy is $m g = \rho V g = \rho \pi r^2 h g$. The text forgot to include the density ρ in computing the mass from the volume), **15**, **17**, **21**. (In **15**, **17**, **21** solve the IVP's only, and indicate whether the configuration is over-damped, under-damped, or critically-damped. Do not graph.)