Math 2250-4
Week 13-14 concepts and homework
sections 7.1-7.4

Section 7.1-7.3 material is due Friday December 6, before class:

7.1: modeling coupled mass-spring systems or multi-component input-output systems with systems of differential equations;converting single differential equations or systems of differential equations into equivalent first order systems of differential equations by introducing functions for the intermediate derivatives; comparing solutions to these equivalent systems.
7.1: 1, 3, **2**, 5, **8**, 11,12, 21, **24**, **25**, **26**.

w13.1) This is related to problems11, 21 ideas above.
a) Consider the IVP for the first order system of differential equations for $x(t)$, $v(t)$:

$$x'(t) = v$$
$$v'(t) = -4x$$
$$x(0) = x_0.$$
$$v(0) = v_0 .$$

Find the equivalent second order differential equation initial value problem for the function $x(t)$.
b) Use Chapter 5 techniques to solve the second order IVP for $x(t)$in part **a**.
c) Use your result from **b** to deduce the solutions $[x(t), v(t)]^T$ for the IVP in **a.**
d) Use your solution formulas for $x(t)$, $v(t)$ from **c** along with algebra and trig identities to verify that the parametric solution curves $[x(t), v(t)]^T$ to the IVP in **a** lie on ellipses in the $x - v$ plane, satisfying implicit equations for ellipses given by

$$x^2 + \frac{v^2}{4} = C, \text{ where } C = x_0^2 + \frac{v_0^2}{4}.$$

e) Reproduce the result of **d** without using the solution formulas, but instead by showing that whenever $x'(t) = v$ and $v'(t) = -4x$ then it must be true also that

$$\frac{d}{dt}\left(x(t)^2 + \frac{v(t)^2}{4}\right) \equiv 0 ,$$

so that $x(t)^2 + \frac{v(t)^2}{4}$ must be constant for any solution trajectory. Hint: use the chain rule to compute the time derivative above, then use the DE's in **a** to show that the terms cancel out.
f) What does your result in **e** have to do with conservation of energy for the undamped harmonic oscillator with mass $m = 1$ and spring constant $k = 4$? Hint: Recall that the total energy for a moving undamped mass-spring configuration with mass m and spring constant k is

$$TE = KE + PE = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

g) Use pplane to create a picture of the tangent field for the first order system of differential equations in this problem, as well as selected solution trajectories. This "phase plane" picture should be consistent with your work above. (And, if you think about orbital "phases" of the moon, you now understand where the terminology "phase plane" comes from.) Print out a screen shot to hand in.

7.2) recognizing homogeneous and non-homogeneous linear systems of first order differential equations; writing these systems in vector-matrix form; statement of existence and uniqueness for IVP's in first order systems of DE's and its consequences for the dimension of the solution space to the first order system, and for the general solution to the non-homogeneous system. Using the Wronskian to check for bases.
7.2: 1, 9, **12**, 13, **14**, **23**.

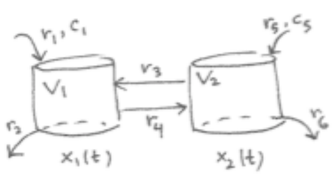
w13.2) This is a continuation of **14**, **23**
a) Use the eigenvalue-eigenvector method of section 7.3 to generate the basis for the general solution that the text told you in **14**.
b) Use pplane to draw the phase portrait for this first order system along with the parametric curve of the solution $[x(t), y(t)]^T$ to the initial value problem in **23**. Print out a screen shot of your work to hand in.

7.3) the eigenvalue-eigenvector method for finding the solution space to homogeneous constant coefficient first order systems of differential equations: real and complex eigenvalues.
7.3: 3, 13, 29, 31, **34**, 36. *In* **34** *you may use technology to find the eigendata.*

w13.3) Use the eigenvalue-eigenvector method (with complex eigenvalues) to solve the first order system initial value problem which is equivalent to the second order differential equation IVP on the Tuesday November 26 notes. This is the reverse procedure from Tuesday, when we use the solutions from the equivalent second order DE IVP to deduce the solution to the first order system IVP. Of course, your answer here should agree with our work there.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

w14.1 Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



a) What equalities between the flow rates guarantee that the volumes V_1, V_2 remain constant?
b) Assuming the equalities in **a** hold, what first order system of differential equations governs the rates of change for $x_1(t), x_2(t)$?

c) Suppose $r_2 = r_3 = r_6 = 100,\ r_1 = r_4 = 200,\ r_5 = 0\ \dfrac{gal}{hour}$; $c_1 = 0.6,\ c_5 = 0\ \dfrac{lb}{gal}$; $V_1 = V_2 = 50\ gal$.
Verify that the constant volumes are consistent with the rate balancing required in **a**. Then show that the general system in **b** reduces to the following system of DEs for the given parameter values:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}.$$

d) Solve the initial value problem for **c**, assuming there is initially no solute in either tank. Hint: Find a particular solution which is a constant vector, and then use $\underline{x} = \underline{x}_P + \underline{x}_H$ to solve the IVP.

e) Check your answer to **d** with technology, and hand in a copy of this verification. For example, in Maple, the "dsolve" command can solve systems of differential equations as well as single differential equations.

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The following material is due Monday December 9, by 5:00 p.m.

7.4) Second order systems of differential equations arising from conservative systems. Identifying fundamental modes and natural angular frequencies; forced oscillation problems and the potential for practical resonance when the forcing frequency is close to a natural frequency.

7.4: **2**, 3, **8**, **12**, 13, 14, **16**, **18** .

w14.2) This is a continuation of **2**, **8**. Now let's force the spring system in problem 2, with a sinusoidal force on the first mass at (variable) angular frequency ω, as in the slightly different text example on pages 440-442. Thus we consider the system

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F_0 \cos(\omega t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

a) Find a particular solution of the form

$$\underline{x}_P(t) = \cos(\omega t)\,\underline{c}.$$

Hint: Plug this guess into the differential equation. You will notice that each term simplifies to some vector times the function $\cos(\omega t)$. Thus, after you factor out the $\cos(\omega t)$ term you are left with a matrix equation to solve for $\underline{c} = \underline{c}(\omega)$. You will get formulas analogous to equations (34, 35) in section 7.5, except your c_1, c_2 will blow up at $\omega = 1, 3$, the natural frequencies for this problem.

b) The general solution to this forced oscillation problem is the particular solution from part (a), plus the general solution to the homogeneous problem, which you found in problem (2). In a physical problem with a slight amount of damping but the same masses and spring constants, the particular solution would be close to the one you found in part (a), and the homogeneous solutions would be close to the ones you found in problem 2, except that they would be (slowly) exponentially decaying because of the damping. Thus the particular solution would be the steady periodic solution, and the homogeneous solution would be transient. By plotting the magnitude $\|\underline{c}(\omega)\| = \sqrt{c_1(\omega)^2 + c_2(\omega)^2}$ as a function of ω, you create a "practical resonance" chart analogous to those we created in Chapter 5. Create such a plot, for the angular frequency range $0 \le \omega \le 5$. Use $F_0 = 4$. Your plot should look like Figure 7.4.10, except your magnitude function will peak at $\omega = 1, 3$.

w14.3) This is a continuation of **18**. In physics you learn that you can recover the final velocities from the initial ones in a conservative problem like 18 by equating the initial momentum $m_1 v_0$ to the final momentum $m_1 v_1 + m_2 v_2$ and the initial kinetic energy $\frac{1}{2} m_1 v_0^2$ to the final kinetic energy $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$, and solving this system of equations for v_1 and v_2 . Carry this procedure out for the data in **18** and show that your answer agrees with your work in that problem.